ECE 340
Probabilistic Methods in Engineering
M/W 3-4:15

Lecture 9: Continuous RVs

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Quiz

- Write down the pmf for a Bernoulli random variable
- What is the relationship between a Bernoulli and a Binomial random variable?
- Write down the pmf for a Binomial random variable?

Reading

- This class: Section 4.1-4.3
- Next class: Section 4.4-4.5

Outline

- Section 4.1-4.3
  - Continuous RV’s
  - CDF
  - PDF
  - Expected value of X

Continuous Random Variables.

As mentioned previously, <u>not all</u> r.v.s are discrete. Today we will study the continuous r.v. (the 2nd general type of random variable) that arises in many applied problems

Continuous Random Variables

A discrete R.V. is one whose possible values either constitute a finite set or else can be listed in an infinite sequence (a list where there is a first element, a second, and so on)

i. Def: A r.v. X is said to be continuous if its set of possible values is an entire interval of numbers

ii. Rule of thumb: <u>before</u> the experiment is run, if you can determine/list all possible values of the random variable, it is a <u>discrete</u> random variable, else it is a <u>continuous</u> random variable.

Continuous Random Variables

i. Let X be a continuous r.v. The probability distribution or probability density function (p.d.f.) of X is a function f(x) [i.e. p(x)] such that for any 2 numbers a and b with a < b

$$P(a \le x \le b) = \int_a^b f(x)dx$$

that is, the probability X takes on a value in the interval [a,b] is the area under the graph ot the density function.

In order that f(x) be a legitimate p.d.f, it must satisfy 2 conditions:

1. f(x) $\geq$ 0, for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Continuous Random Variables

i. If X is a continuous r.v., then for any number c, $P(X=c) = f(c) = 0$

   furthermore, for any 2 numbers a and b with $a < b$ $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$

   The probability assigned to any particular value is zero and the probability of an interval does not depend on whether or not its end points is included.

ii. Note: Unlike in discrete distribution, the distribution of a continuous r.v. usually cannot be derived from simple probabilistic reasoning.

   Instead, one must make a choice of a p.d.f based on prior knowledge and available data. Fortunately, there are general/common families of pdf's. That has been found to fit well in a variety of experimental situations.

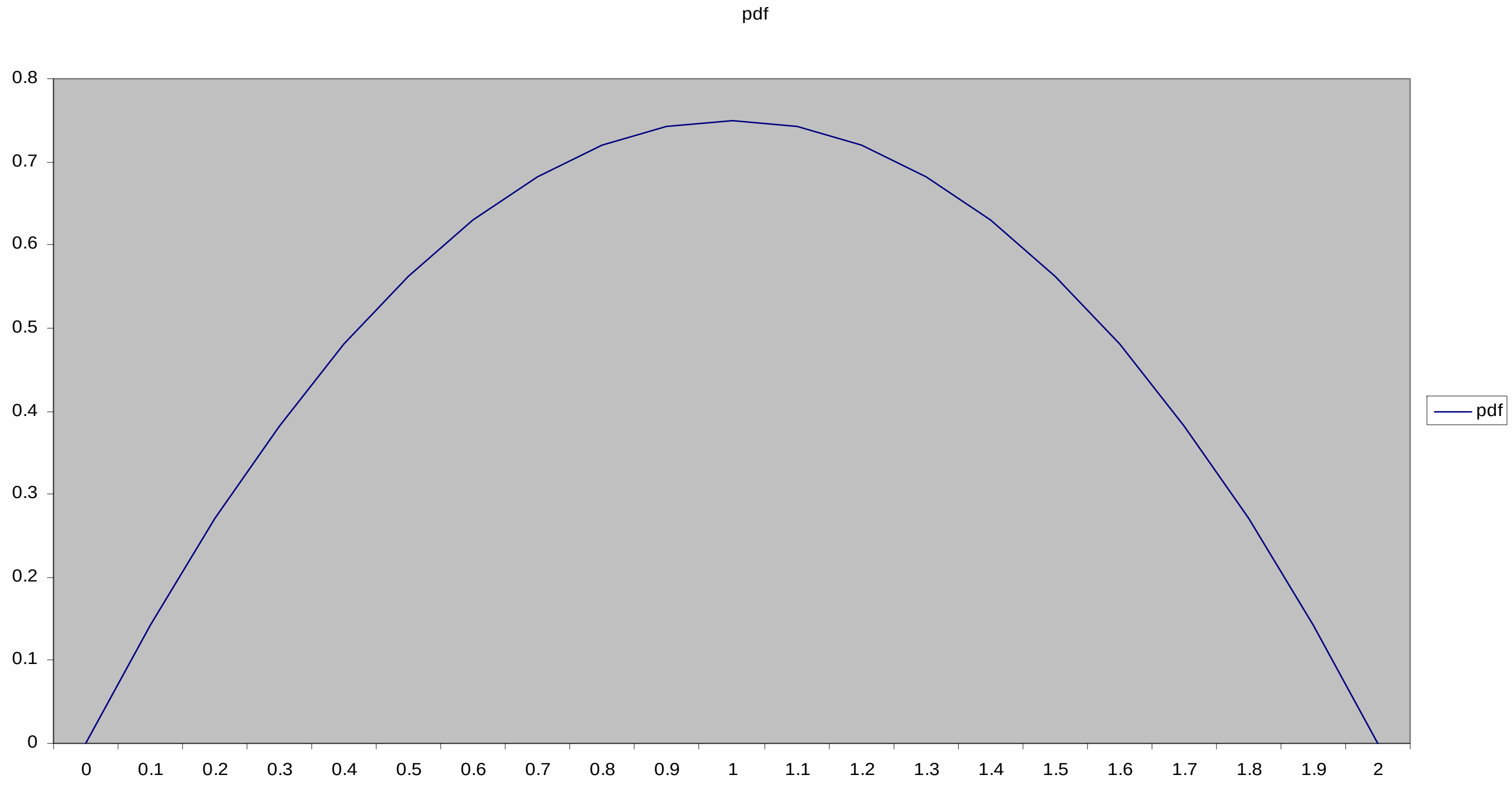
Continuous Random Variable

- An example would be the following random variable, distributed as follows:

$$P(X = x) = \left\{ \begin{array}{l} -\frac{3x^2}{4} + \frac{3x}{2}; 0 \le x \le 2 \\ 0; otherwise \end{array} \right\}$$

Continuous Random Variable

- **Its density function would be:**



Continuous Random Variables

In the discrete case, the p.m.f is obtained from taking the difference between 2 F(X) [cumulative] values ∴P(X=a) = F(a) – F(a – 1)

The continuous analogue of a difference is a derivative.

∴If X is a continuous r.v. with a pdf f(x) and c.d.f F(X), then at every x at which the derivative F'(X) exists

$$\frac{\partial F(x)}{\partial x} = f(x) \qquad or \qquad F'(x) = f(x)$$

Probability Density Function pdf

$$\frac{dF_X(a)}{dx} \equiv f_X(a)$$

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$$

$$\lim_{a\to\infty} F_X(a) = \lim_{a\to\infty} \int_{-\infty}^{a} f_X(x)\,dx \quad \Rightarrow \quad \boxed{1 = \int_{-\infty}^{+\infty} f_X(x)\,dx}$$

Probability Density Function pdf

- **For a continuous random variable X,**

  $$P[a \le x \le b] = P[a < x \le b] = F_X(b) - F_X(a)$$

using $$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$$

$$F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$$

$$F_X(b) - F_X(a) = \cancel{\int_{-\infty}^{a} f_X(x)\,dx} + \int_{a}^{b} f_X(x)\,dx - \cancel{\int_{-\infty}^{a} f_X(x)\,dx}$$

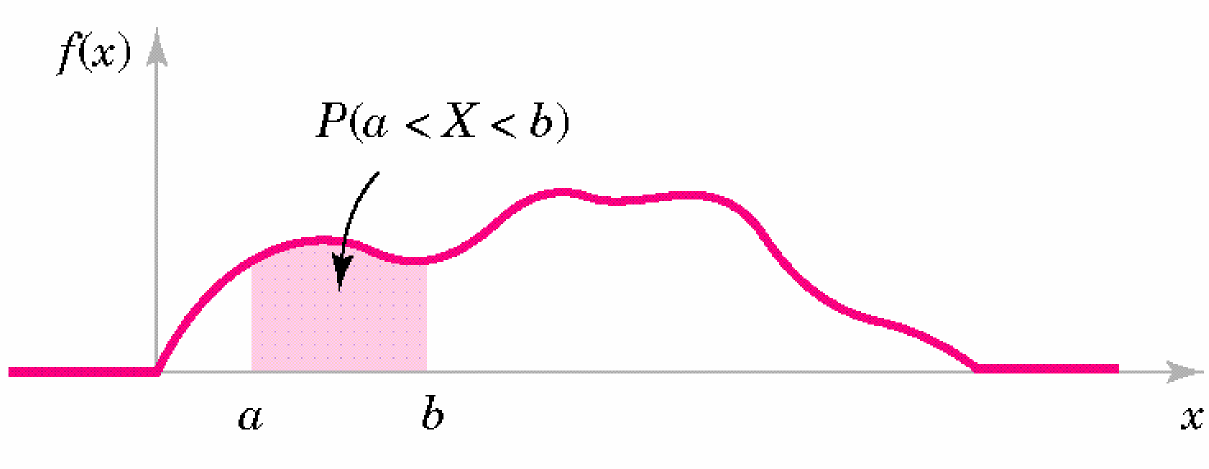
Probability Density Function pdf

- Probability that a continuous random variable X takes on values between a and b:

$$\boxed{P[a \le X \le b] = \int_a^b f_X(x)\,dx}$$

Probability Density Function

$$P(a < X < b) = \text{area under } f(x) \text{ from } a \text{ to } b = \int_a^b f(x)\,dx \qquad \text{(3-2)}$$



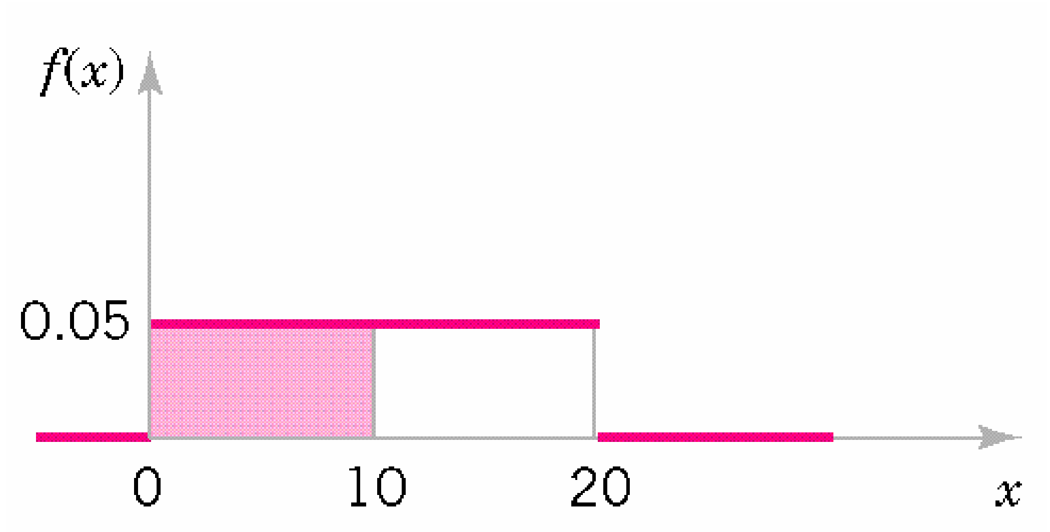
If X is a continuous random variable, then for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$$

Uniform

X = current measured in a thin copper wire (in mA)

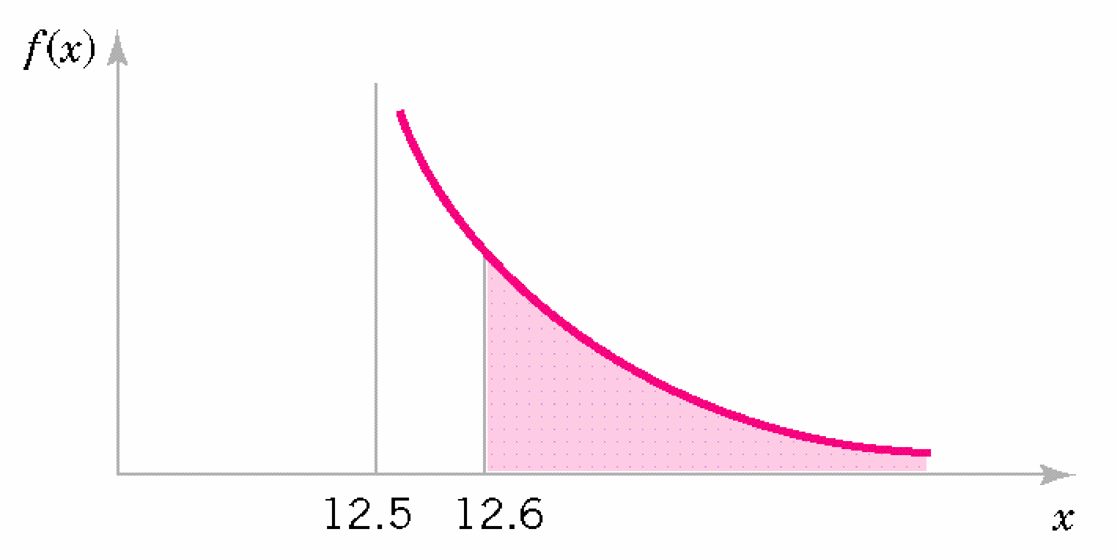
The PDF of X is given by f(x) = 0.05, 0 < x < 20.



- Find
  - P(X < 8)
  - P(X < 8 / X > 6)

Exponential

X = diameter (mm) of hole drilled in a sheet metal component



- Find
  - P(X > 12.6)
  - P(X < 14 / X >12.6)

Cumulative Distribution Function

Several of the important concepts introduced in the study of discrete distributions are equivalent with continuous distributions by replacing summation with integration.

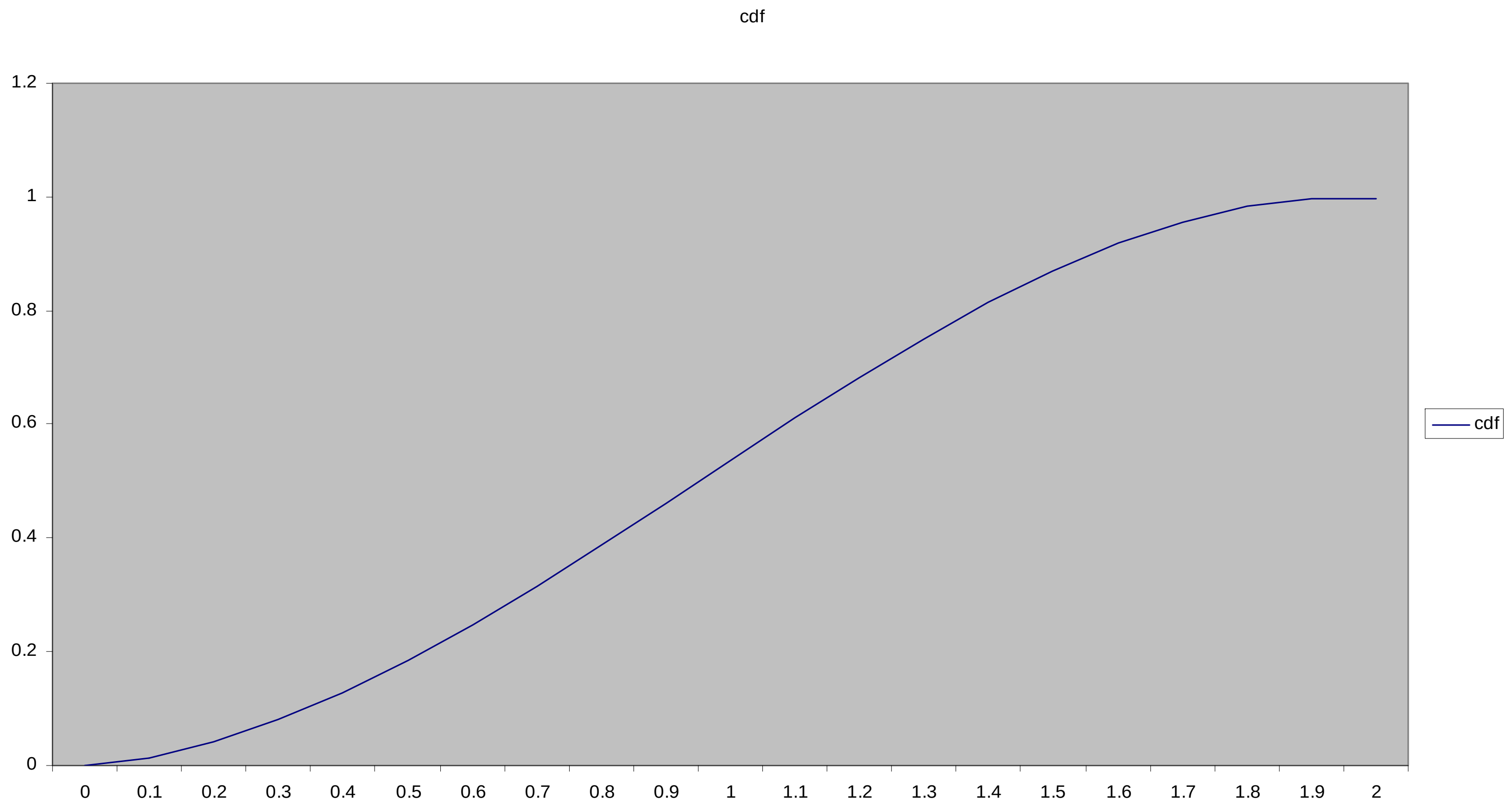
Cumulative Distribution Function (cdf)

- cdf $F_X(a)$ of a random variable X is defined as "the probability that X has a value smaller than or equal to a":

$$F_X(a) = P[X \le a] \qquad \forall\ -\infty \le a \le +\infty$$

Cumulative Distribution Function

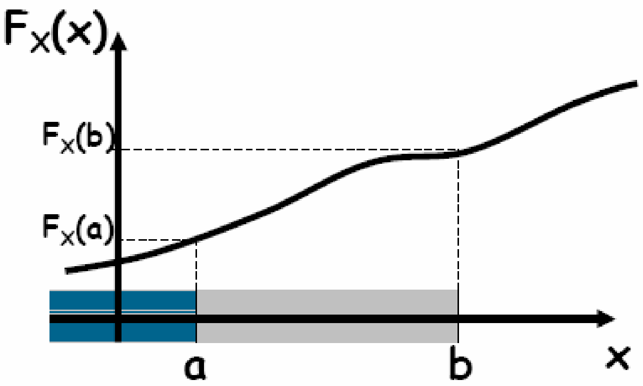
- The c.d.f. for the previous example is shown below:



Properties of cdf $F_X(a)=P[X \leq a]$

- If $a \leq b \Rightarrow F_X(a) \leq F_X(b)$

  i.e. F_X is a non-decreasing function

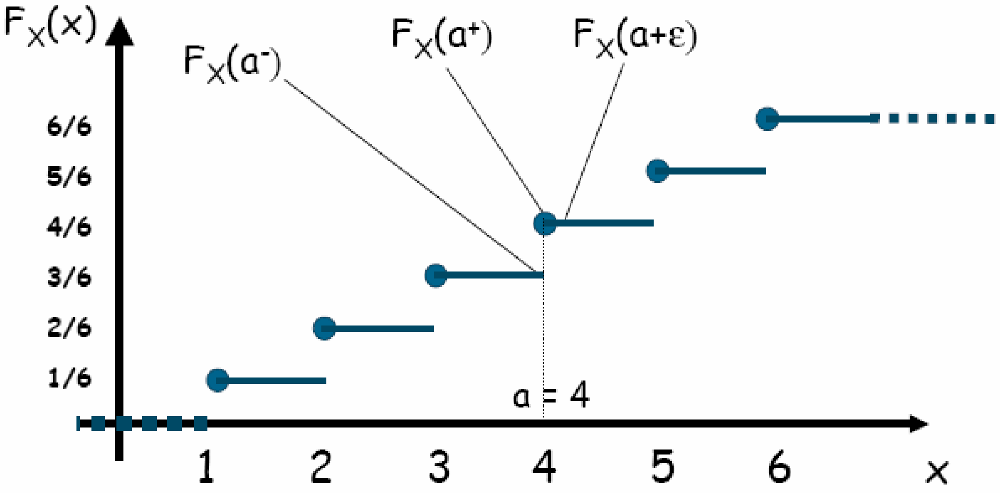


Properties of cdf $F_X(a)=P[X \le a]$

- $0 \le F_X(a) \le 1$
- $\lim_{a \to \infty} F_X(a) = 1$
- $\lim_{a \to -\infty} F_X(a) = 0$

(Dis)Continuity Properties of cdf

- For any positive ε, $\lim_{\varepsilon \to 0} F_X(a+\varepsilon) = F_X(a^+)$
- F_X is "continuous from the right"

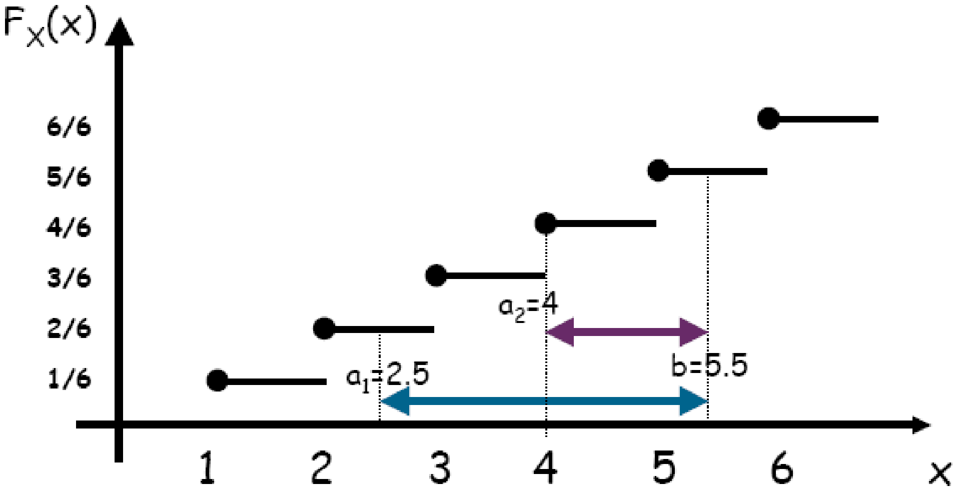


Properties of cdf

- $P[a < X \leq b] = F_X(b) - F_X(a)$

$P[2.5 = a_1 < X \leq b = 5.5] = (5/6) - (2/6) = 3/6$

$P[4 = a_2 < X \leq b = 5.5] = (5/6) - (4/6) = 1/6$

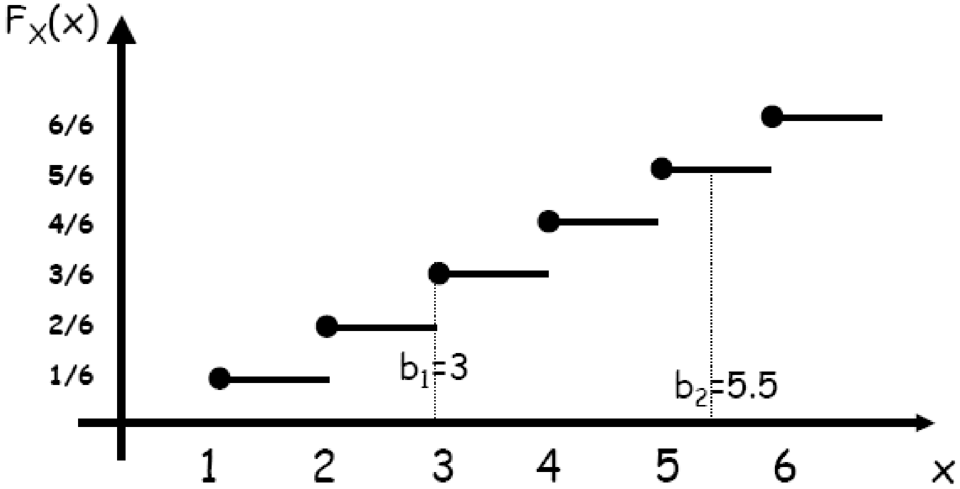


Properties of cdf

- **$P[X=b] = F_X(b) - F_X(b^-)$**

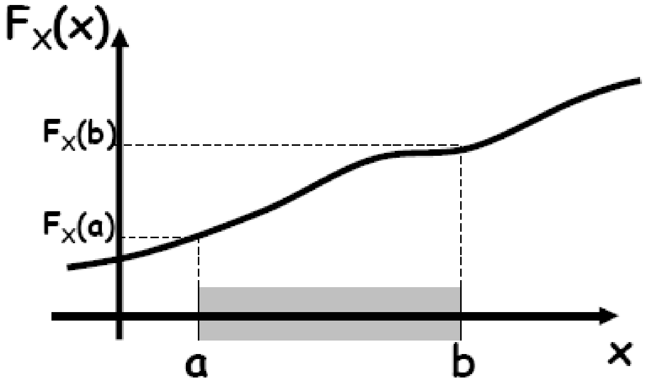
$P[X=b_1=3] = (3/6) - (2/6) = 1/6$

$P[X=b_2=5.5] = (5/6) - (5/6) = 0$



Properties of cdf

- For a continuous random variable X, the following probabilities are equivalent:
  - $P[a \le X \le b]$
  - $P[a < X \le b]$
  - $P[a \le X < b]$
  - $P[a < X < b]$



Expected Value for a Continuous R.V.

a. For the discrete r.v. X, E[X] was obtained by summing xp(x) over possible x values. Here we replace summation by integration and the p.m.f by pdf to get a continuous weighted average.

   i. Def: The expected value or mean value of a continuous r.v. X with pdf f(x) is:

$$\mu_x = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Expected Value for a Continuous R.V.

i. Def: If X is a continuous r.v. with pdf f(x) and h(x) is any function of X, then:

$$\mu_{h(x)} = E[h(x)] = \int_{-\infty}^{\infty} h(x) f(x) dx$$

ii. Note: The rules for evaluating functions of random variables are valid for the use with continuous random variables
E[aX + b] = aE[X] + b

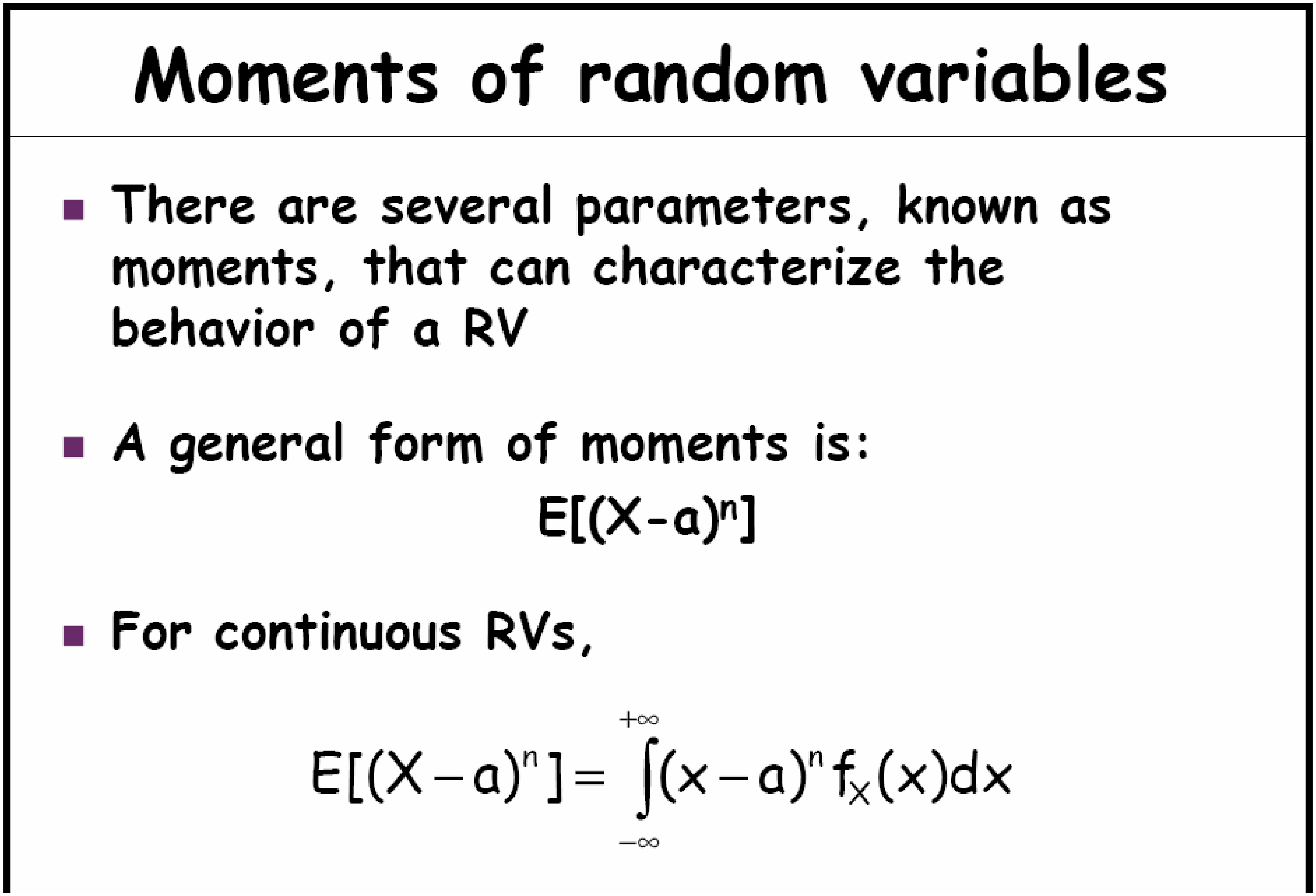
Expected Value for a Continuous R.V.

- Continuing with the previous example distribution, and integrating from the allowed values from 0 to 2, the expected value is:

$$P(X=x)=\begin{Bmatrix} -\frac{3x^2}{4}+\frac{3x}{2};0\leq x\leq 2 \\ 0;otherwise \end{Bmatrix}$$

$$E(x)=\int_{x=0}^{2}-\frac{3x^3}{4}+\frac{3x^2}{2}dx$$

$$E(x)=\left(\frac{-3x^4}{16}+\frac{x^3}{2}\right)\Bigg|_0^2=-3+4=1.0$$



Moments of random variables

- There are several parameters, known as moments, that can characterize the behavior of a RV
- A general form of moments is:

$$E[(X-a)^n]$$

- For continuous RVs,

$$E[(X-a)^n] = \int_{-\infty}^{+\infty} (x-a)^n f_X(x) dx$$

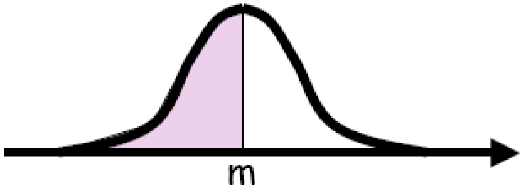
Moments of random variables

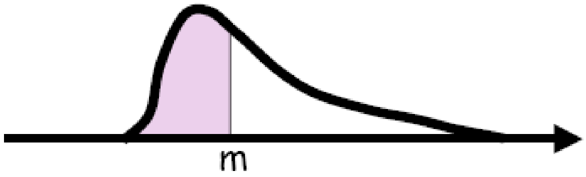
- **Special cases of $E[(X-a)^n]$:**
  - The mean: $E[X] = m$, $a = 0$, $n = 1$
  - The second moment: $E[X^2]$, $a = 0$, $n = 2$
  - The central moments: $E[(X-m)^n]$, $a = m = E[X]$
  - The variance: $E[(X-m)^2]$, $a = m = E[X]$, $n = 2$

The Mean

- **The mean is the “expected value” or the “average value” of a RV**

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)dx$$





The mean represents the “center of gravity” for the probability (mass) density function

The Second Moment

- **The second moment $E[X^2]$ is the (total) power of the random variable X**

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx$$

The second moment $E[X^2]$ represents the “moment of inertia with respect to the origin” for the probability (mass) density function

The Second Moment

- **The second moment E[X²] can be expressed as:**

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx = \int_{-\infty}^{+\infty} (x-m+m)^2 f_X(x)dx$$

$$= m^2 \cancelto{1}{\int_{-\infty}^{+\infty} f_X(x)dx} + \int_{-\infty}^{+\infty} (x-m)^2 f_X(x)dx + 2m \cancelto{0}{\int_{-\infty}^{+\infty} (x-m) f_X(x)dx}$$

$$= m^2 + \int_{-\infty}^{+\infty} (x-m)^2 f_X(x)dx$$

The Second Moment

- Therefore, the second moment $E[X^2]$ is the sum of two “power” terms:

$$E[X^2] = \underset{\text{“DC” Power}}{\underset{\swarrow}{m^2}} + \underbrace{\int_{-\infty}^{+\infty} (x-m)^2 f_X(x)dx}_{\text{“AC” Power}}$$

The Variance

- The variance σ^2 is the second central moment:

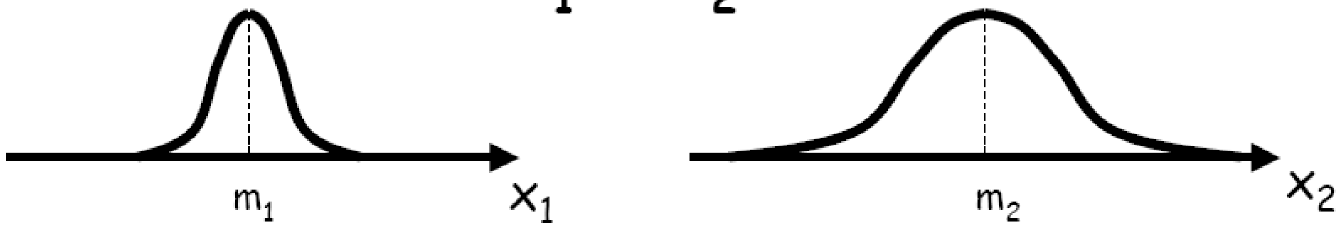
$$\sigma^2 = E[(X-m)^2] = \int_{-\infty}^{+\infty} (x-m)^2 f_X(x) dx$$

The variance represents the "central moment of inertia"

The Variance

- The variance σ^2 is a measure of "the amount of variation of a RV around its mean"
- σ is defined as the "standard deviation"

$\sigma_1 < \sigma_2$



The Variance

- The variance can be computed from the second moment (total power) and the first moment ("DC" power):

$$\sigma^2 = E[(X - m)^2] = E[X^2] - m^2$$

Variance of a Continuous R.V.

Define: The variance of a continuous random variable X with pdf f(x) with mean μ is:

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E[(x-\mu)^2]$$

and the standard deviation of X is

$$\sigma_X = \sqrt{V(X)}$$

The easiest way to compute σ^2 is to compute

$$V(X) = E[X^2] - (E[X])^2$$

Variance of a Continuous R.V.

- **Continuing as before, and integrating from 0 to 2 as before, the variance of the distribution is:**

$$P(X = x) = \left\{ \begin{matrix} -\frac{3x^2}{4} + \frac{3x}{2}; 0 \leq x \leq 2 \\ 0; otherwise \end{matrix} \right\}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = \int_{x=0}^{2} x^2 f(x) dx - 1.0^2$$

$$V(x) = \int_{0}^{2} -\frac{3x^4}{4} + \frac{3x^3}{2} dx - 1.0^2$$

$$V(x) = \left(\left(-\frac{3x^5}{20} + \frac{3x^4}{8} \right) \Bigg|_0^2 \right) - 1.0^2 = 1.20 - 1.00 = 0.20$$

Variance of a Continuous R.V.

As in expected values, the rules of variance learned in discrete distribution are valid in the continuous case.

$$V(aX + b) = a^2V(X)$$

Back to Moments $E[X^n]$

- Can the mean and the variance completely specify a random variable (e.g. its pdf)?

  In general, NO

- Knowledge of all moments $E[X^n]$, n=1, 2, …, (if they exist) can be used to uniquely determine the pdf of the RV X

NORMAL (GAUSSIAN)

- The most important continuous distribution in probability and statistics
- The story of the outcome of normal is really the story of the development of statistics as a science.
- Gauss discovered this while incorporating the method of least squares for reducing the errors in fitting curves for astronomical observations.

PDF OF NORMAL

Definition

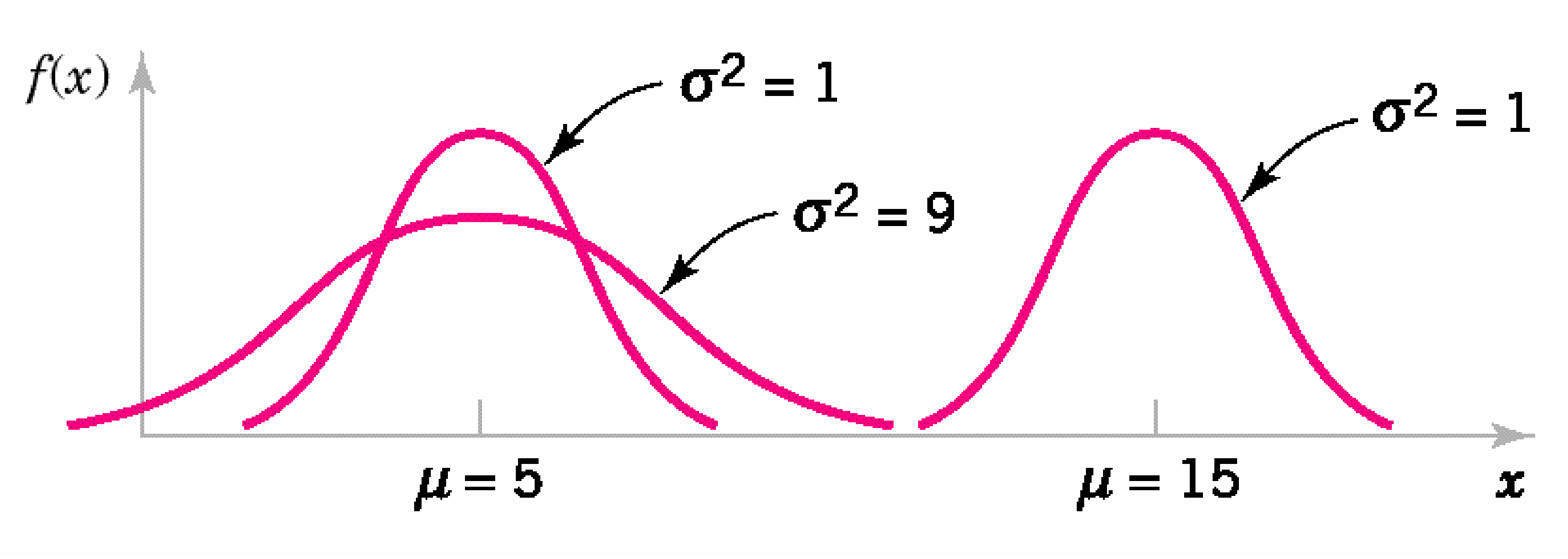
A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad -\infty < x < \infty \qquad (3\text{-}4)$$

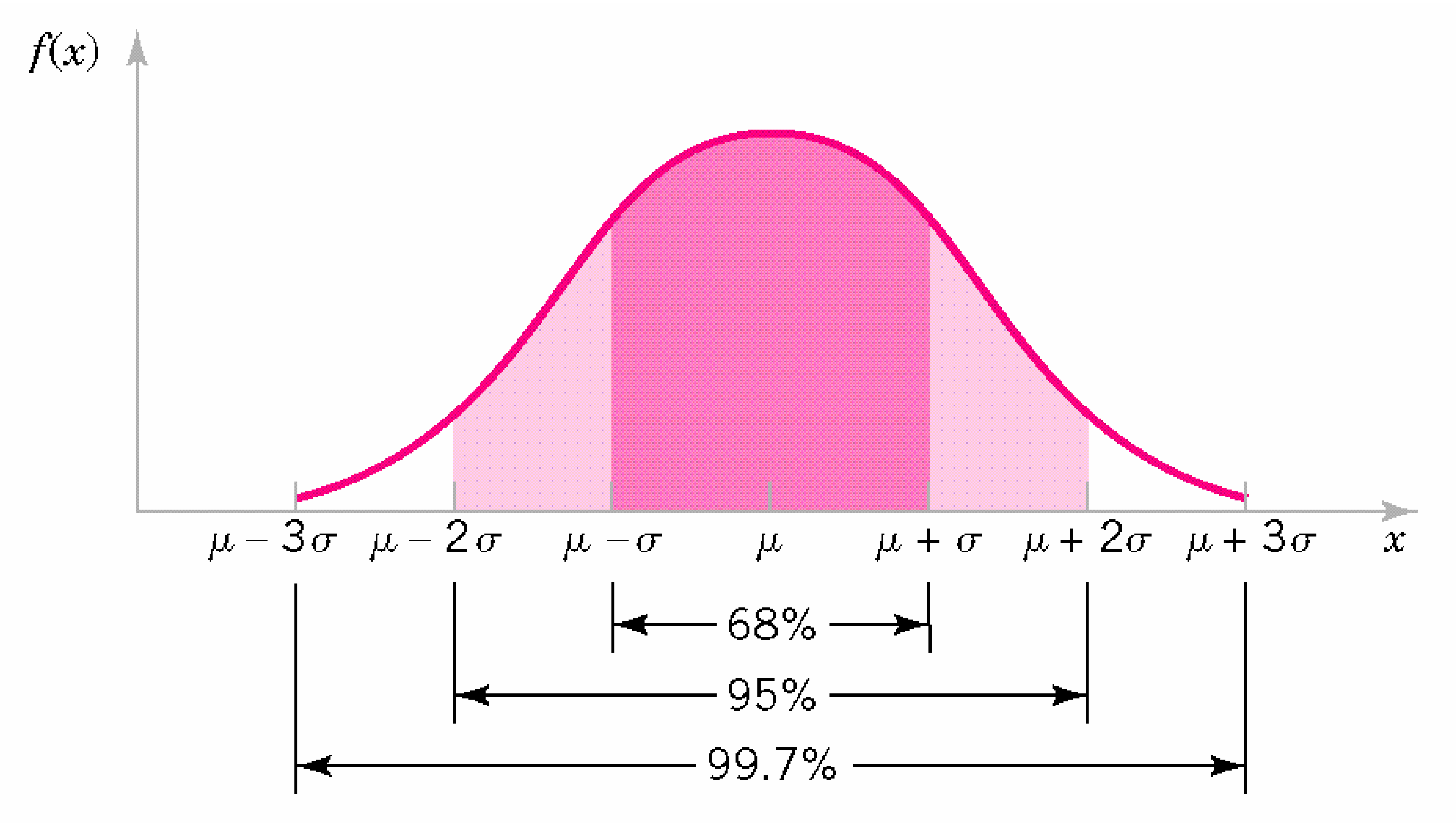
has a **normal distribution** with parameters μ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \qquad \text{and} \qquad V(X) = \sigma^2$$

Graphs of various normal PDF



$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

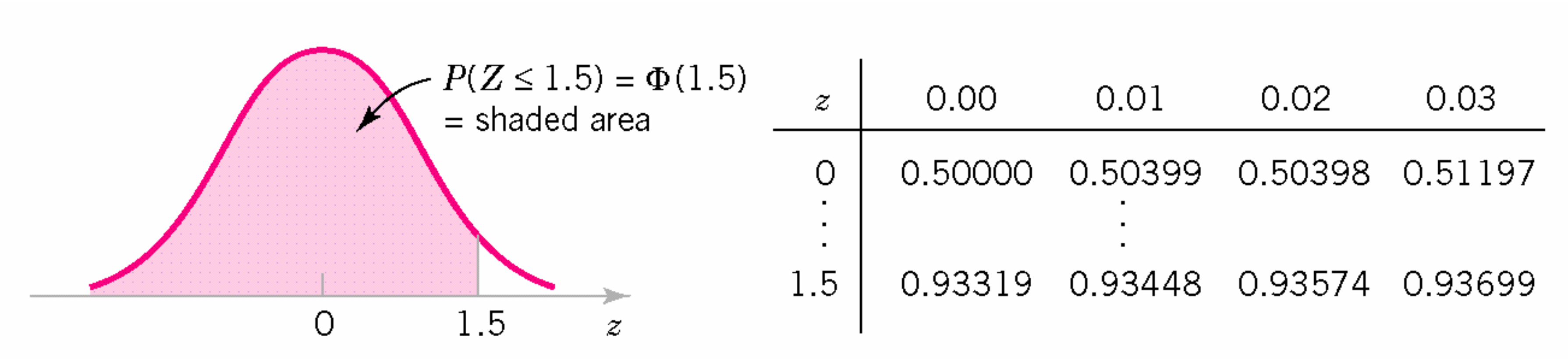


Definition

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **standard normal random variable.** A standard normal random variable is denoted as Z.

Definition

The function $\Phi(z) = P(Z \leq z)$ is used to denote a probability from Appendix Table I. It is called the cumulative distribution function of a standard normal random variable.

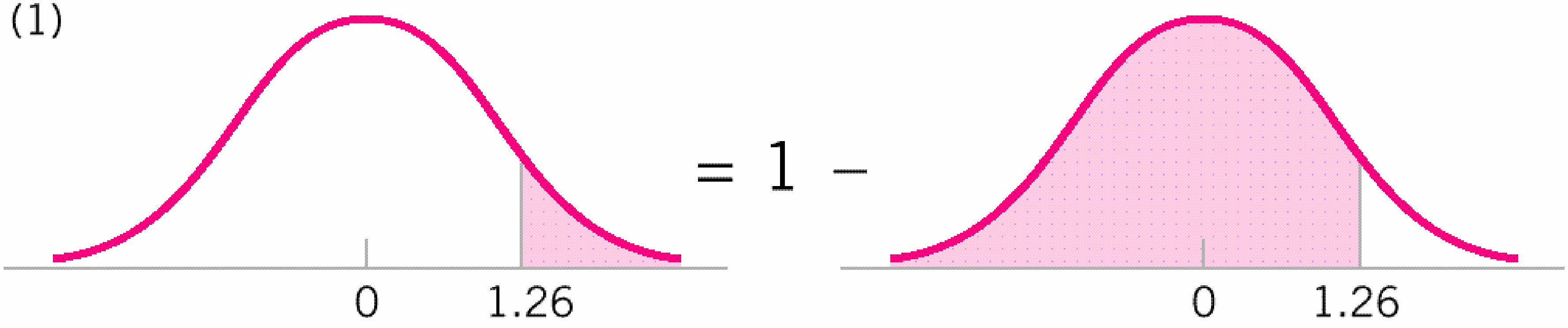


z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

CALCULATION OF NORMAL PROBABILITIES

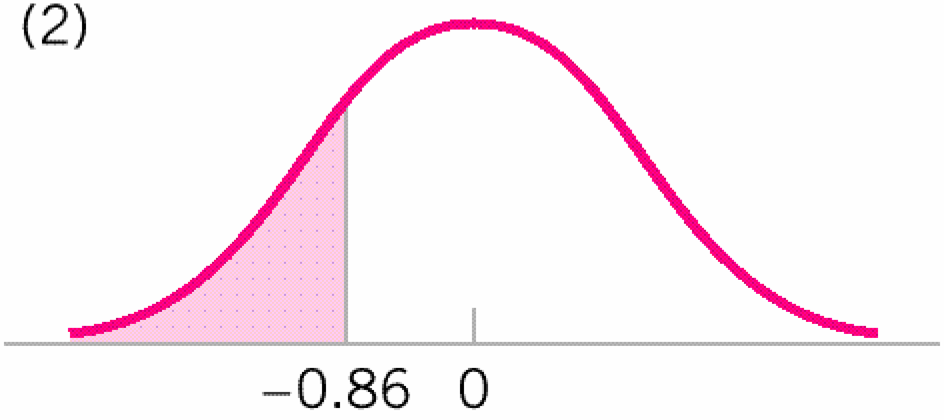
(1) $P(Z > 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.89616 = 0.10384$

(1)



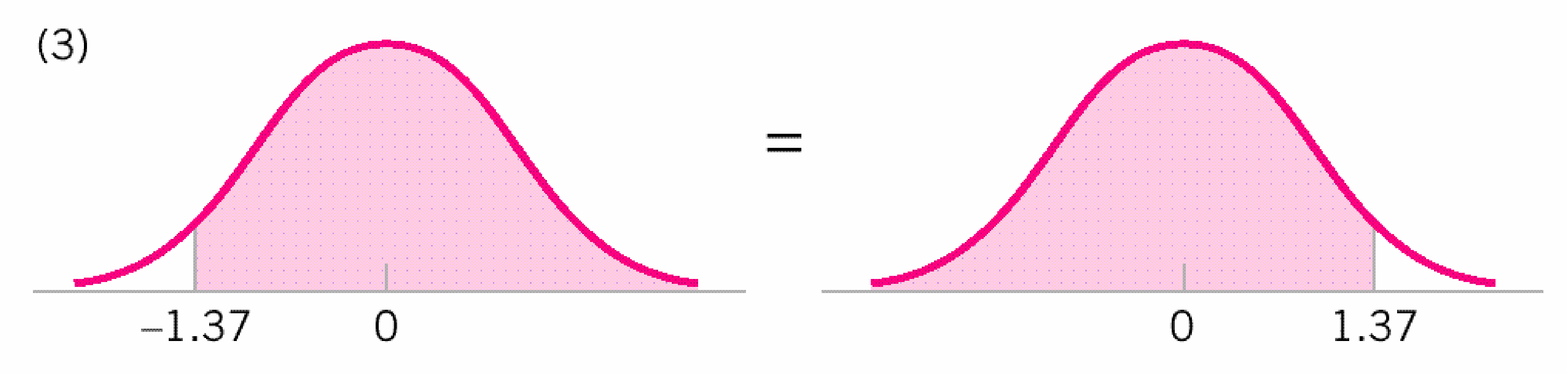
(2) $P(Z < -0.86) = 0.19490.$

(2)

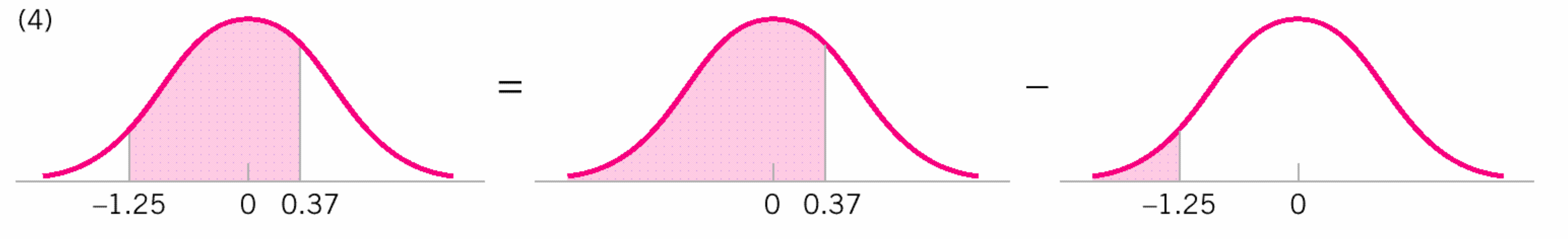


EXAMPLE (cont’d)

(3) $P(Z > -1.37) = P(Z < 1.37) = 0.91465$

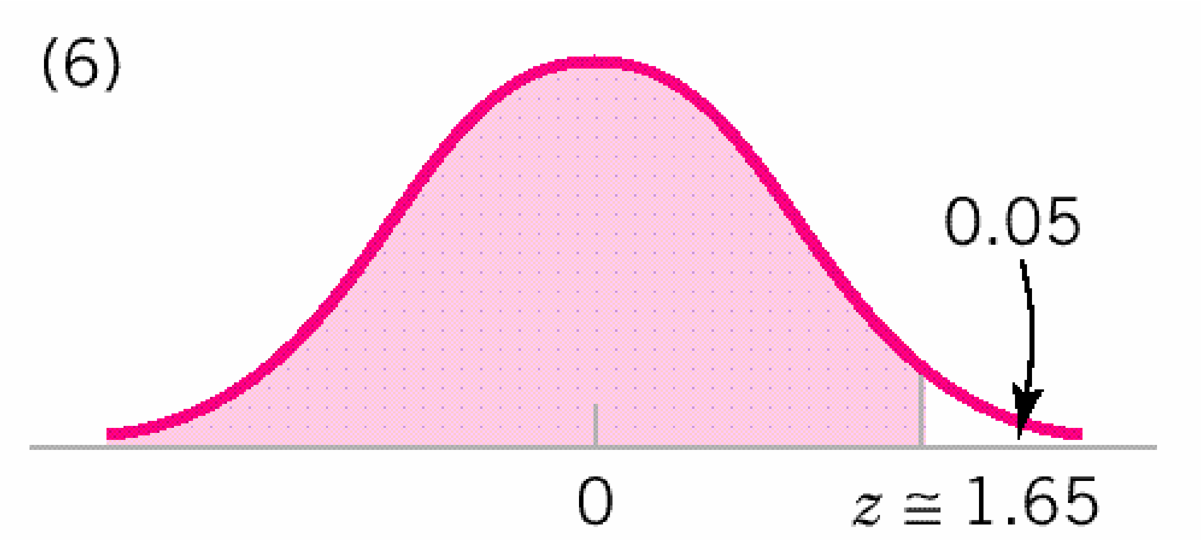


(4) $P(-1.25 < Z < 0.37)$

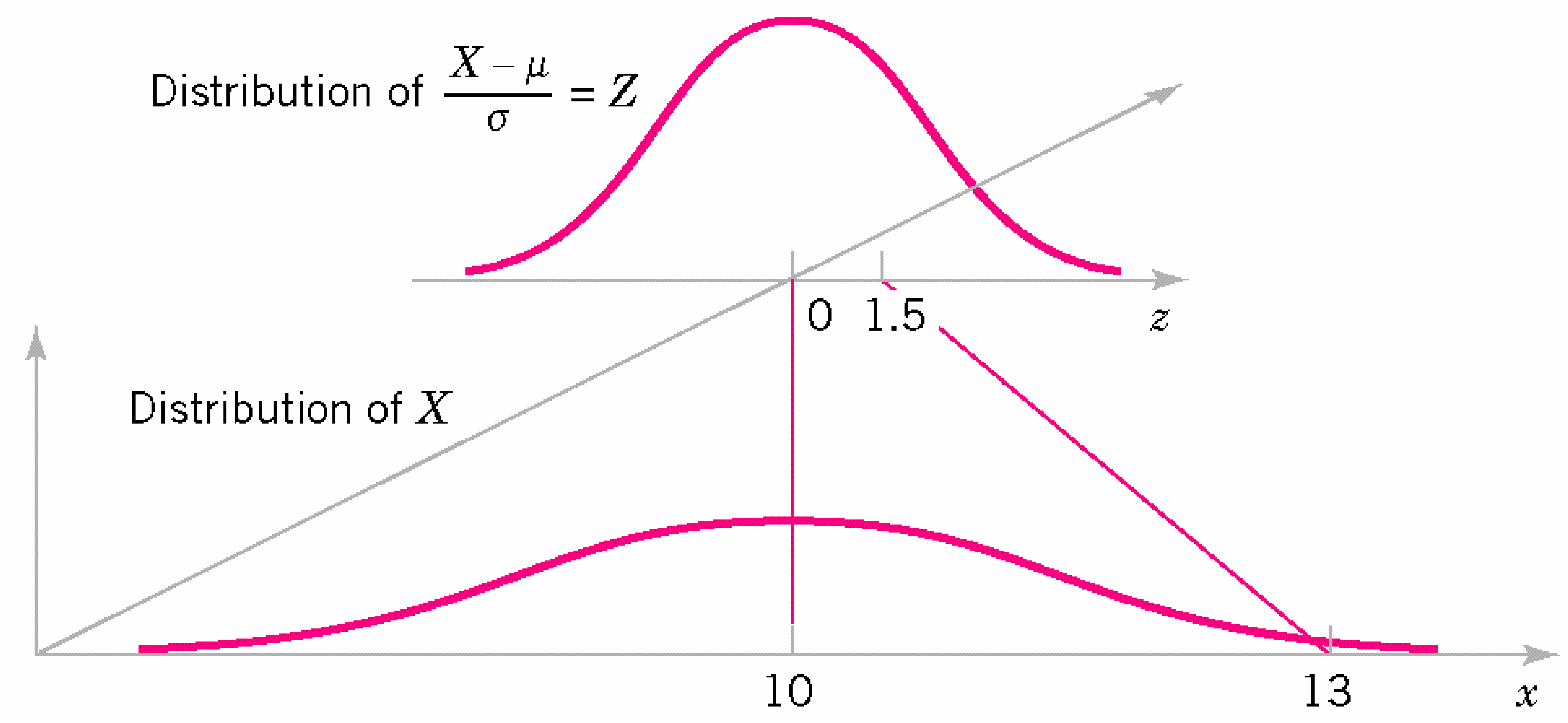


EXAMPLE (cont’d)

(6) $P(Z > ???) = 0.05$ or $P(Z < ???) = 0.95$



How to standardize?



Standardize

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) \qquad (3\text{-}5)$$

where

Z is a **standard normal random variable,** and

$z = (x - \mu)/\sigma$ is the ***z*-value** obtained by **standardizing** X.