ECE 340

Lecture 7: Variance & conditional pmf

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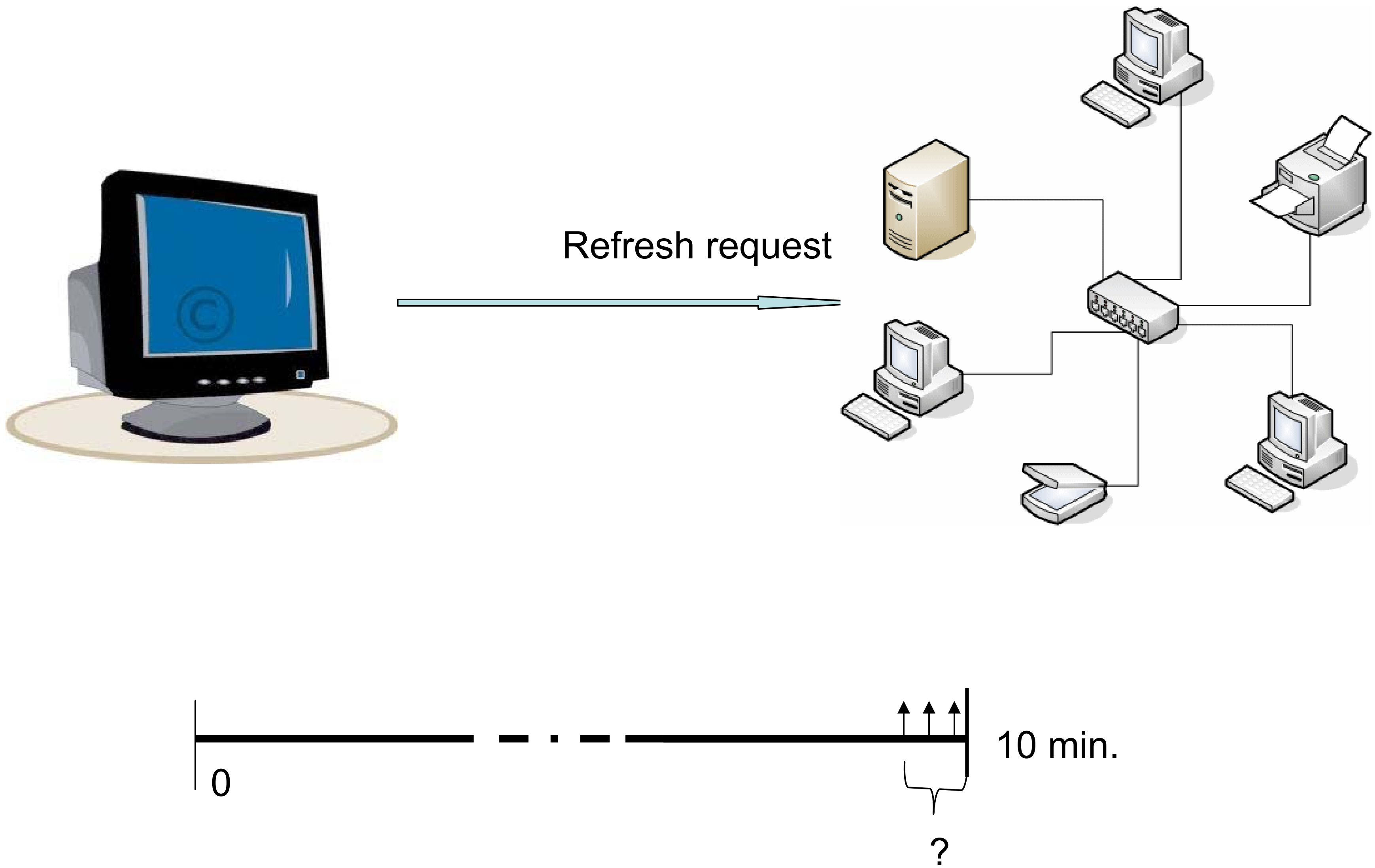
Example to review

Problem 3.18

A computer reserves a path in a network. To extend the reservation the computer must successfully refresh message before the expiry time. The messages are lost with probability ½. Suppose that it take 10 sec. to send a refresh message and receive acknowledgement. When should the computer start sending a refresh messages in order to have a 99% chance of successfully extending the reservation time.


Refresh request
0
10 min.
?

- Let X be the times that a refresh request needs to be sent till successfully received.
- S_X is {1,2,3,…}
- Let Y be the seconds that a refresh request needs to be sent till successfully received.
- S_Y is {10,20,30,…}
- Success rate of a message is 1/2

What is the set B, which begins with 1 and increases accordingly, has 99% chance of successfully extending reservation?

pmf of X?

$$p_X(x_k) = p(x = K) = (\frac{1}{2})^{k-1} \cdot (\frac{1}{2}) = (\frac{1}{2})^k$$

Set B probability ?

$$P[B] = \sum_{k \in B} p(x_k) = \sum_{k=1}^{N} (\frac{1}{2})^k$$

P[B]>99%

$$p[B] \geq \frac{99}{100}$$

$$\sum_{k=1}^{N} \frac{1}{2}^k \geq \frac{99}{100}$$

$$\frac{0.5(1 - 0.5^n)}{1 - 0.5} \geq \frac{99}{100}$$

$$\frac{1}{100} \geq 0.5^n$$

$$n \geq 7$$

- N=7
- 70 seconds before expiry time
- What is the expected time (seconds) that it takes to renew the reservation?

$$E_7[Y] = \sum_{k=10}^{70} y \cdot p_X(X_k)$$

$$= 10 \cdot (\frac{1}{2})^1 + 20 \cdot (\frac{1}{2})^2 + 30 \cdot (\frac{1}{2})^3 + ... + 70 \cdot (\frac{1}{2})^7$$

$$= 19.30$$

Quiz 3

Reading

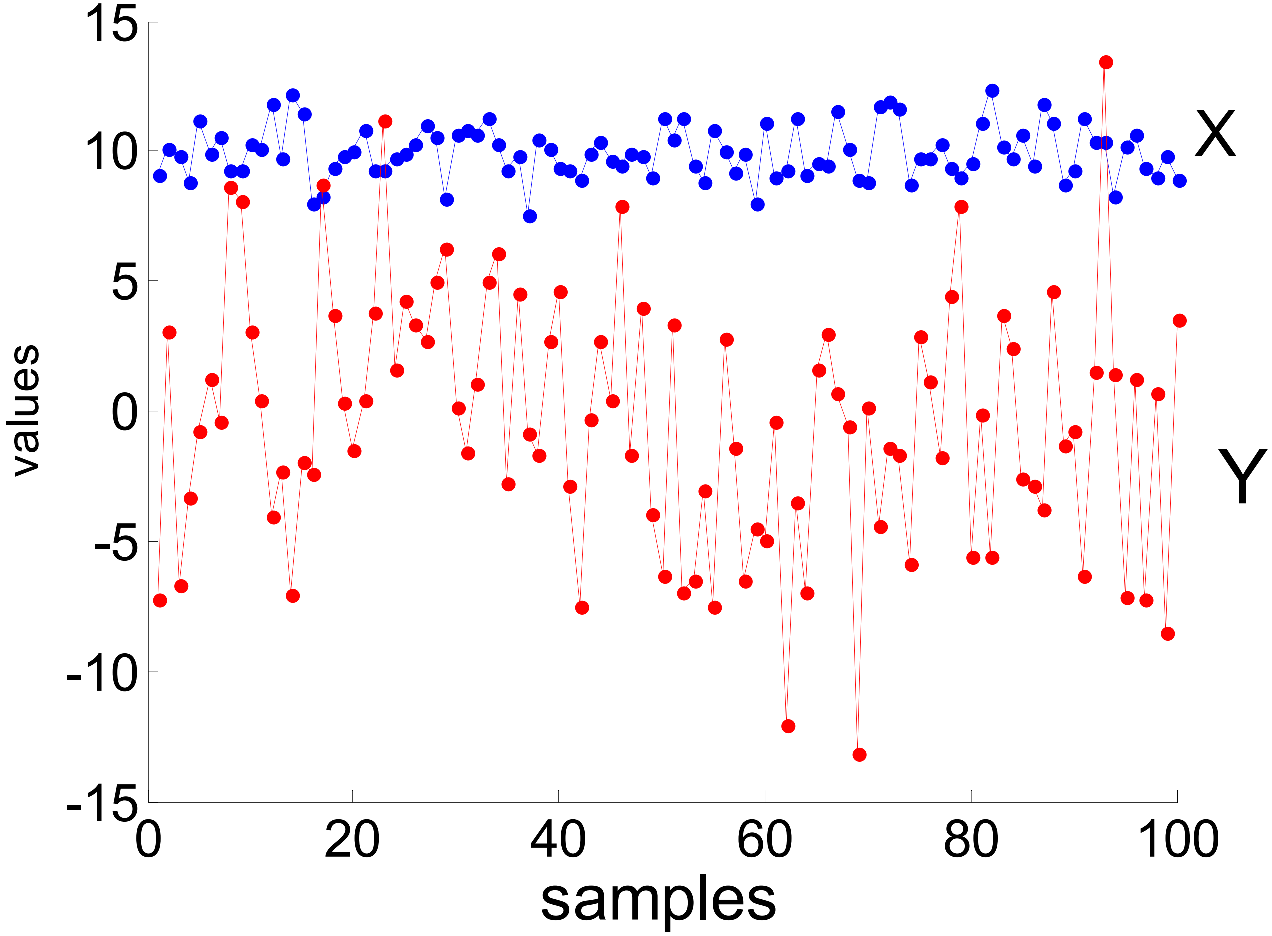
- This class: Section 3.3, 3.4
- Next class: Section 3.5

Outline

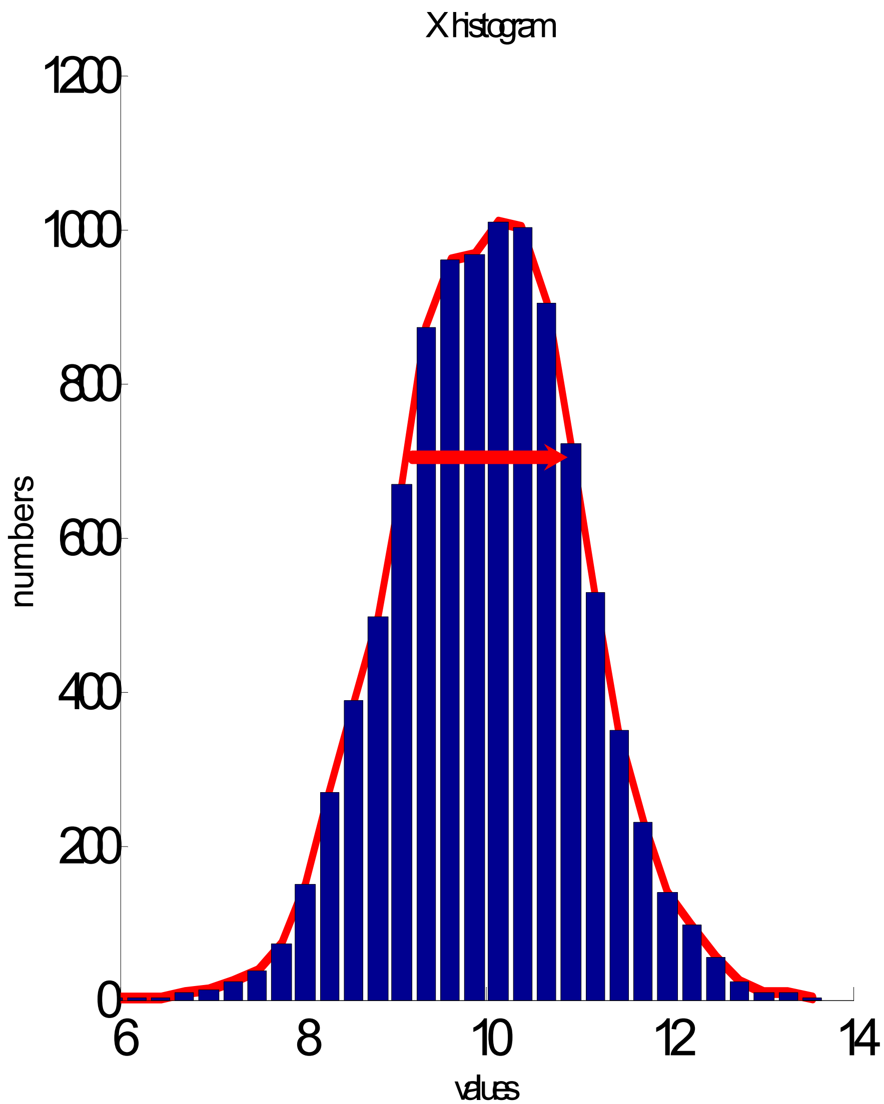
- Variance of a random variable
- Moment of a random variable
- Conditional Probability mass function

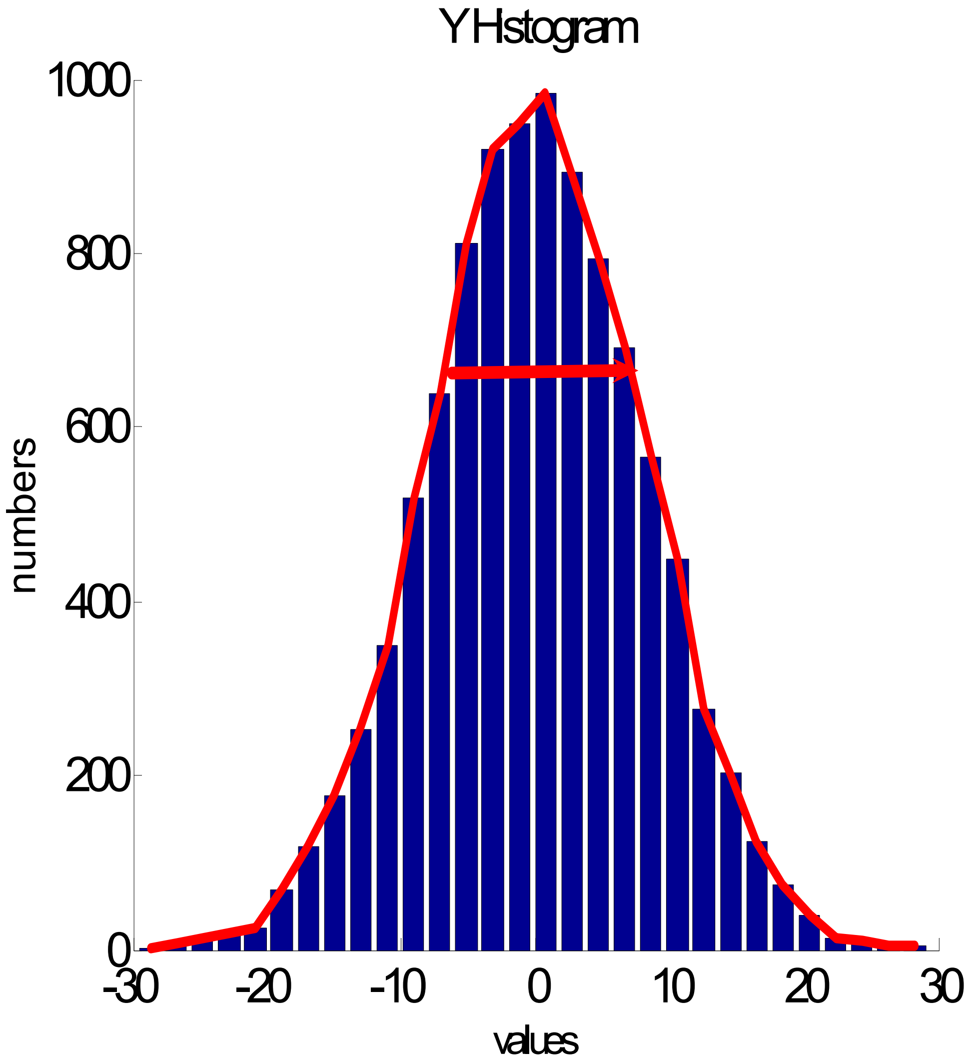
- Samples

Random variable X and Y



Histogram of variable X and Y





- Variance

$$\sigma^2 = VAR[X] = E[(X - m_x)^2]$$

where, m_x is the expected value of X.

$$\sigma^2 = E[(X - m_x)^2] = \sum_{x \in S_X} (x - m_x)^2 \cdot p_x(x)$$

- Standard deviation

$$\sigma = STD[X] = VAR[X]^{1/2}$$

- $VAR[X]=E[(x-m_x)^2]$

$=E[x^2-2xm_x+m_x^2]$

$=E[x^2]-2m_xE[x]+m_x^2$

$=E[x^2]-m_x^2$

$E[x^2]$ is the second moment of X

$E[x^n]$ is the nth moment of X

Example 1

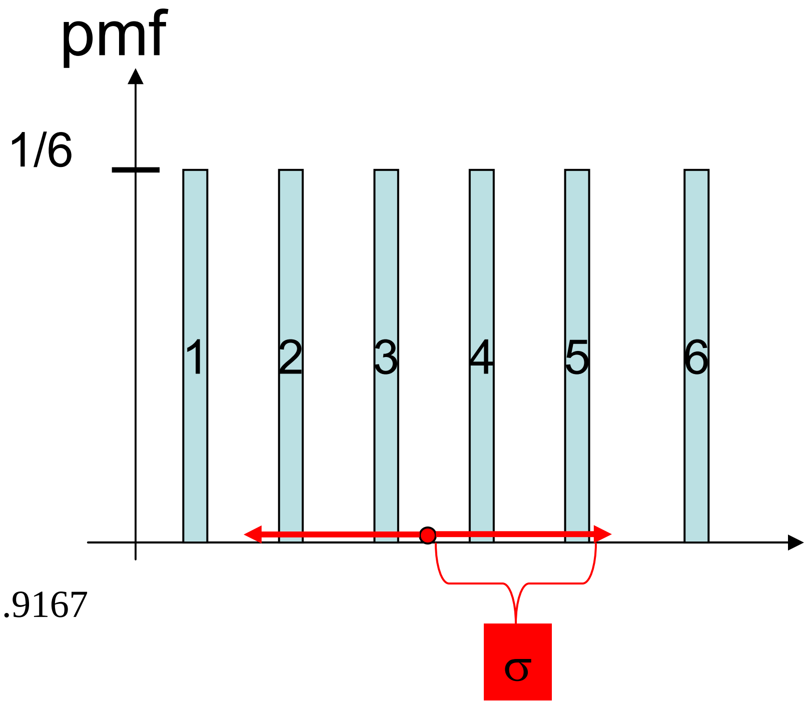
- Uniform random variable
- VAR[X] =?
- STD[X]

E[X]=3.5

$$VAR[X] = \sum_{k=1}^{6} (k-3.5)^2 \cdot \frac{1}{6} = 2.9167$$

$$VAR[X] = (\sum_{1}^{6} k^2 \cdot \frac{1}{6}) - 3.5^2 = \frac{91}{6} - 12.25 = 2.9167$$

$$STD[X] = 1.707$$



Example2

- Let X be the number of heads in three tosses of a fair coin

- $VAR[X]=E[(X-m_x)^2] = 3/4$

- $VAR[X]=E[X^2]- m_x^2 = 3/4$

1) X=c

VAR[X] = ? $E[(x-c)^2]=0$

2) Y=X+c;

$$VAR[Y]=VAR\ [X+c]= E[\ (x+c-\ (E[X]+c)\ )^2\ ]$$
$$=E[\ (\ x-E[X]\ )^2\ ]$$
$$=E\ [\ (x-\ m_x)^2\ ]$$
$$=VAR[X]$$

3) Y=cX;

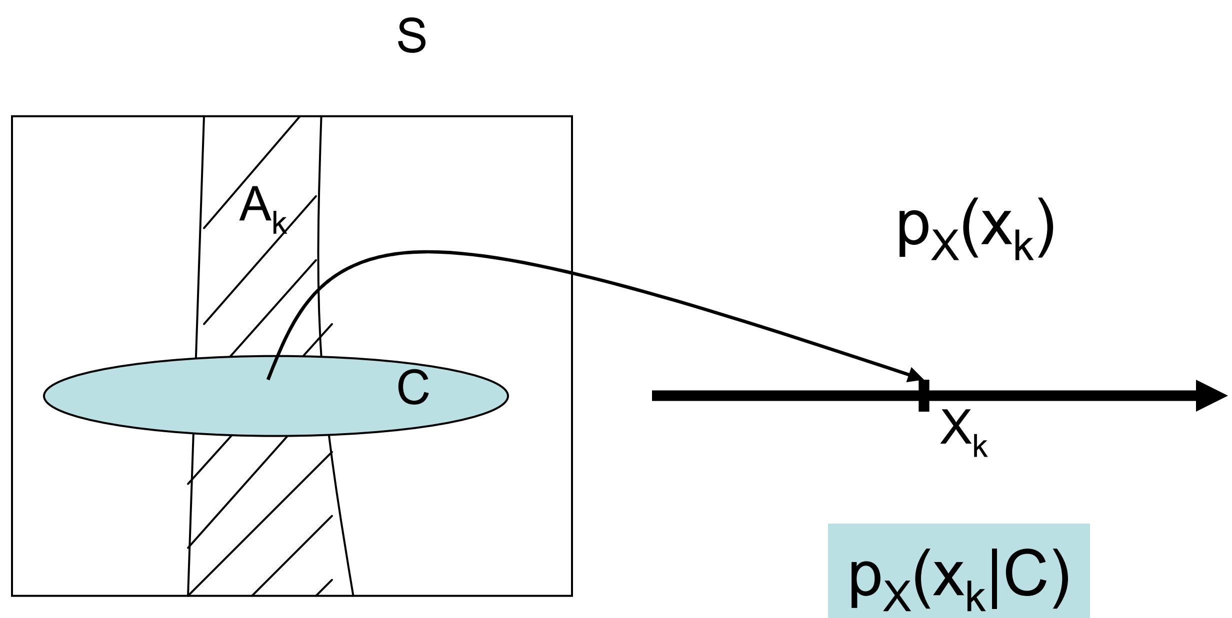
$$VAR[Y]=VAR\ [cX]= E[\ (cx-\ cE[X]\ )2\ ]$$
$$=E\ [c^2(\ x-E[X]\ )^2\ ]$$
$$=c^2\ E\ [\ (x-\ mx)^2\ ]$$
$$=\ c^2\ VAR[X]$$

Conditional probability mass function

- Let X be a discrete random variable with a pmf $p_X(x)$. Let C be an event with P[C]>0.

The conditional probability mass function of X : $p_X(x|C) = P[\, X=x \mid C]$

$$= \frac{P[\{X=x\} \cap C]}{P[C]}$$


S
A_k
C
p_X(x_k)
x_k
p_X(x_k|C)

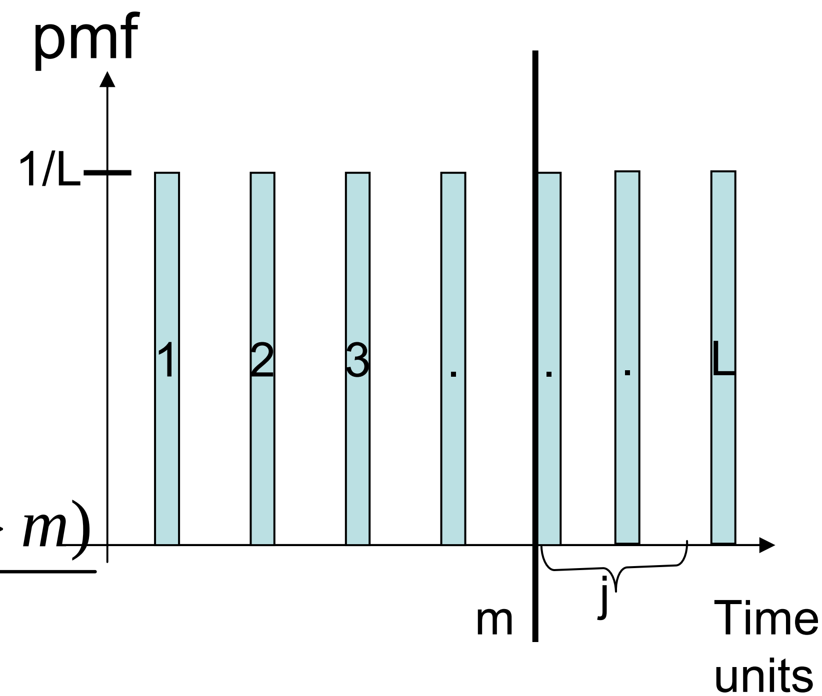
Example 3

Residual waiting time

Let X be the time required to send a message, where X is the a uniform random variable with Sx {1,2,3,…,L}. Suppose a message has been transmitting for m time units, find the probability that remaining transmission time is j time units.

- Given X>m; Let C={X>m}
- P[X=m+j]; $L \geq m+j > m$

$p_x(X=m+j \mid C)$

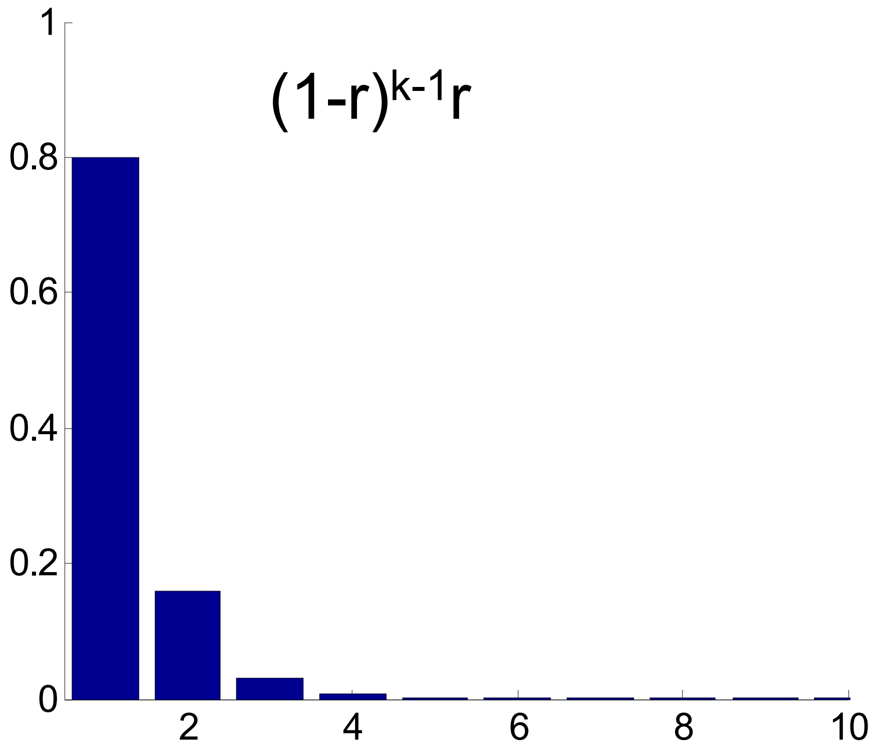


$$p_x(x = m + j \mid C) = \frac{p(x = m + j \cap x > m)}{p[C]}$$

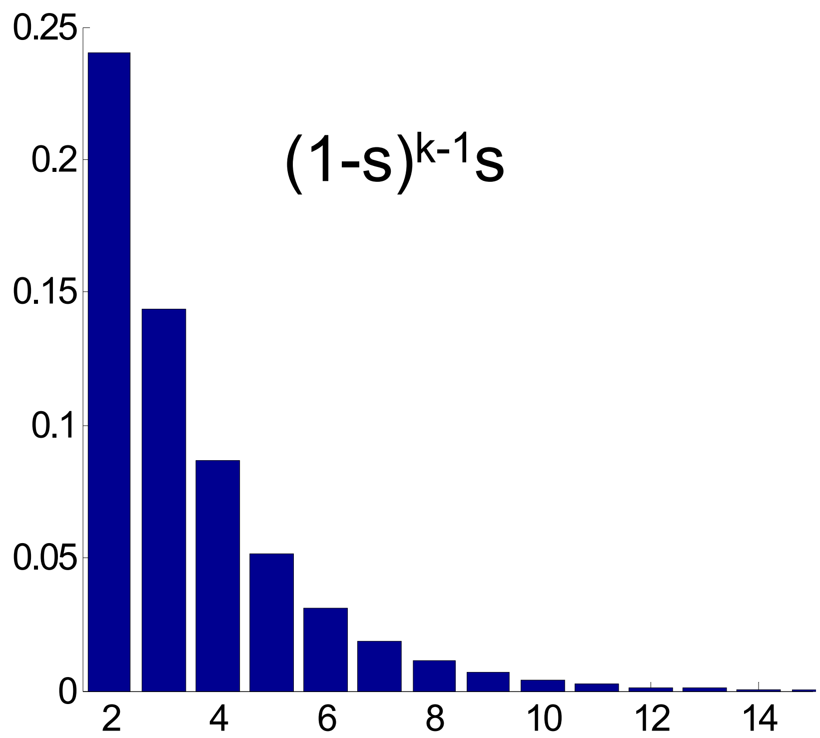
$$= \frac{1/L}{(L-m)/L} = \frac{1}{L-m}$$

Example 4 -Device lifetime

- Type1 (a)



Type 2 (1-a)



$$p_X(x) = \sum_{i=1}^{n} p_X(x \mid B_i)p[B_i]$$

$$\sum_{i=1}^{n} B_i = S_X; B\mathrm{i} \text{ are disjoint events}$$

$$p_X(x) = p_X(x_k \mid B_1)p[B_1] + p_X(x_k \mid B_2)p[B_2]$$
$$= (1-r)^{k-1}r\alpha + (1-s)^{k-1}s(1-\alpha)$$

Conditional expected value

- E[X|B]

$$E[X \mid B] = \sum_{x \in S_X} x \cdot p_X(x \mid B)$$

Conditional variance

- VAR[X|B]

$$VAR[X \mid B] = E[(X - m_{x|B})^2 \mid B]$$

$$= \sum_{k=1}^{\infty} (x_k - m_{x|B})^2 \cdot p_X(x_k \mid B)$$

$$= E[X^2 \mid B] - m_{x|B}{}^2$$

HW 3

- problems 3.11, 3.20, 3.21, 3.31
- Due date: Feb. 20 at the beginning of class