ECE 340

Sequential experiments

Lecture 5

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Reading

- This class: Section 2.6
- Next class: Section 3.1 & 3.2

outline

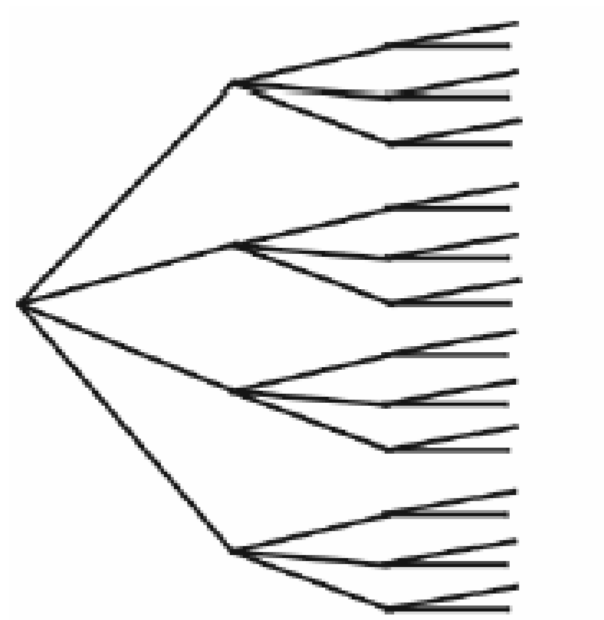
- Sequential experiments
  - Sequences of independent experiment
    - Binomial probability law
    - Multinomial probability law
    - Geometric probability law
  - Sequences of dependent experiments
- Examples

- **Sequential experiments**

Often the main experiment consists of a sequence of sub-experiments.

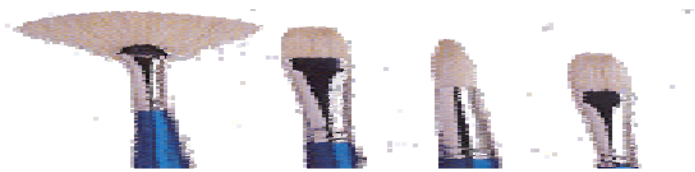
EX: Toss coin n times 

EX: A project contains N steps,

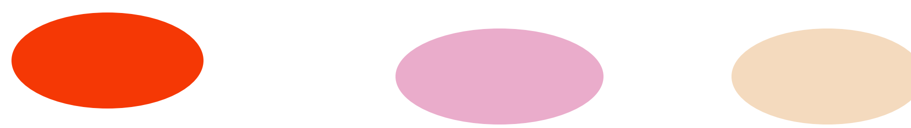


Sequences of independent experiment

- Paint a painting of the sun
- 1) select the canvas {small, medium, large}
- 2) select the brush {fan, round, flat, filbert}



- 3) select the color {red, pink, yellow }



- Sample space:
- {small, medium, Large} X {fan, round, flat, filbert} X {red, pink, yellow}

$P[\text{Large canvas} \cap \text{Fan brush} \cap \text{Yellow}]$

Assume sub-experiments are independent

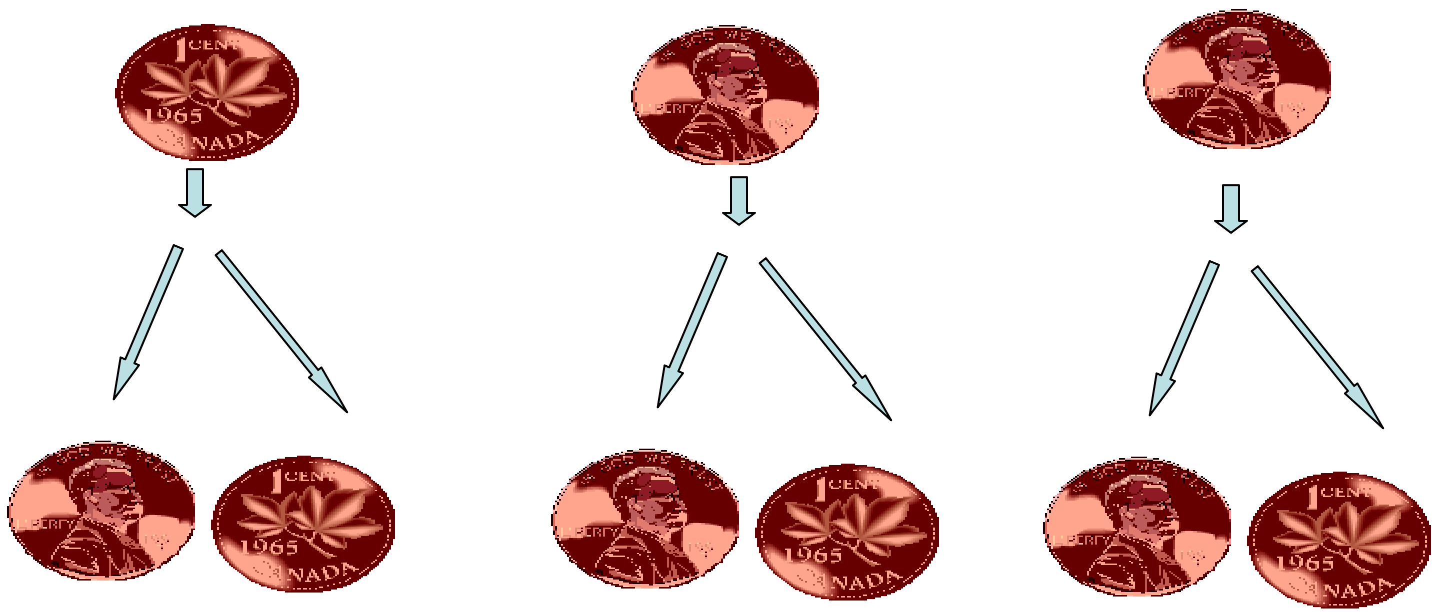
$$P[L \cap F \cap Y] = P[L]\, P[F]\, P[Y]$$

General form:

$$P[A1 \cap A2 \cap A3 \ldots An] = P[A1]\, P[A2] \ldots P[An]$$

Example1
Sequences of independent experiment

- Flip coin 3 times experiment
- Each sub-experiment is independent. The probability of heads is ρ



- Sample Space:{ {H,T } X {H,T} X {H,T} }

- $P[HHH]=P[H]P[H]P[H]=\rho^3$
- $P[HHT]=P[H]P[H]P[T]=\rho^2(1-\rho)$
- $P[HTH]=P[H]P[T]P[H]=\rho^2(1-\rho)$
- $P[THH]=P[T]P[H]P[H]=\rho^2(1-\rho)$
- $P[TTH]=P[T]P[T]P[H]=\rho^1(1-\rho)^2$
- $P[THT]=P[T]P[H]P[T]=\rho^1(1-\rho)^2$
- $P[HTT]=P[H]P[T]P[T]=\rho^1(1-\rho)^2$
- $P[TTT]=P[T]P[T]P[H]=(1-\rho)^3$

Summary

- k is the # of heads,
- $P[k=3] = \rho^3$
- $P[k=2] = 3\rho^2(1-\rho)$
- $P[k=1] = 3\rho^1(1-\rho)^2$
- $P[k=0] = (1-\rho)^3$

$$p_3(k) = \binom{3}{k} p^k (1-p)^{3-k} \qquad \text{for } k=0,1,3$$

Binomial probability law

Bernoulli trial:
perform an experiment, the outcome is 'success' or 'fail'

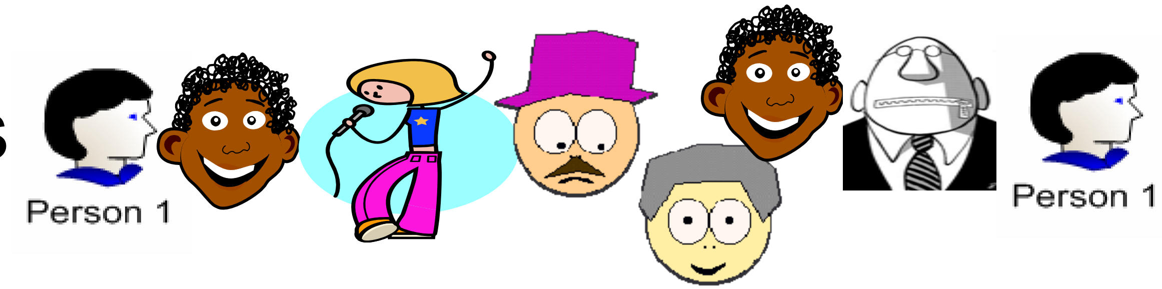
Repeat the experiment N times, each time the probability of success is P[A]=p.
Then, K successes in N Bernoulli trials is ?

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } k=0,1,...n$$

$$\binom{\mathrm{n}}{\mathrm{k}} = \frac{n!}{k!(n-k)!}$$

Example 2

- Eight speakers



- Let k be the number of active speakers. A speaker is active with probability 1/3.
- P[K>6] ?

$$P[k=7]+P[k=8]$$

$$=\binom{8}{7}\frac{1}{3}^{7}\,\frac{2}{3}^{1}+\binom{8}{8}\frac{1}{3}^{8}$$

$$=8\,\frac{1}{3}^{7}\,\frac{2}{3}^{1}+\frac{1}{3}^{8}=0.00259$$

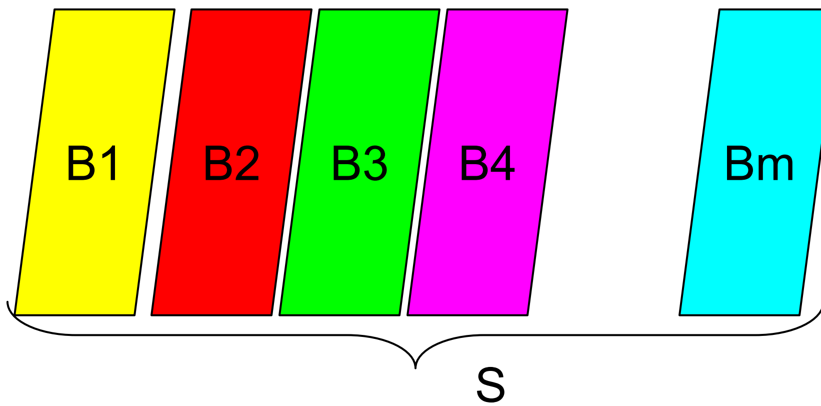
- Binomial probability law
  - When to apply ?

  - How to apply ?

Multinomial Probability law

- Binomial : two events 'Success' or 'fail'
- Multinomial: M exclusive events' existence

$$P[Bi] = p_i; \qquad p_1 + p_2 + \ldots p_m = 1$$

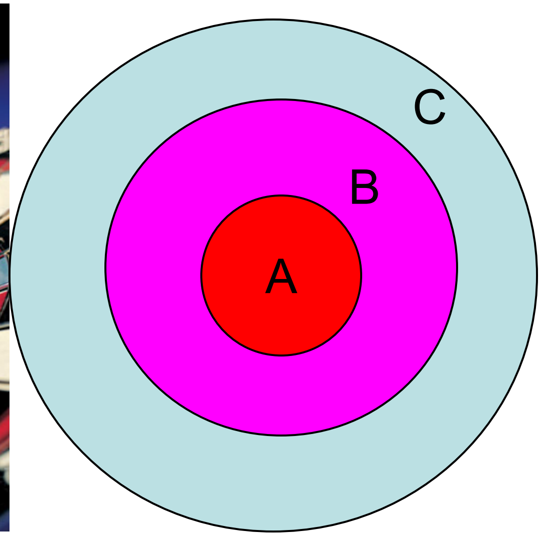


- N repetitions of experiment, k_i is # of B_i occurs
- P[(k1,k2,k3…km)] ? Where, K1+K2+..Km=N

Multinomial Probability law

$$P[(k_1, k_2, ..., k_m)] = \frac{n!}{k_1!k_2!...k_m!} p_1^{k_1} p_2^{k_2} ... p_m^{k_m}$$

Example 2



P[A]=0.2
P[B]=0.3
P[C]=0.5

- Throw a dart **NINE** times, what is probability that the dart landed in AreaA **3** times, AreaB **3** time, and AreaC **3** times?

- P[3,3,3]=?

$$P[3,3,3] = \frac{9!}{3! \cdot 3! \cdot 3!} 0.2^3 0.3^3 0.5^3 = 0.04536$$

Binomial vs. Multinomial

- Independent
- M=2 (Success/ Failure)
- P[S]=p; P[F]= 1-p
- Repeat N time

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for k=0,1,...n

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Independent
- M=m (B1, B2, B3…Bm)
- P[Bi]=pi ; (Σpi=1)
- Repeat N time

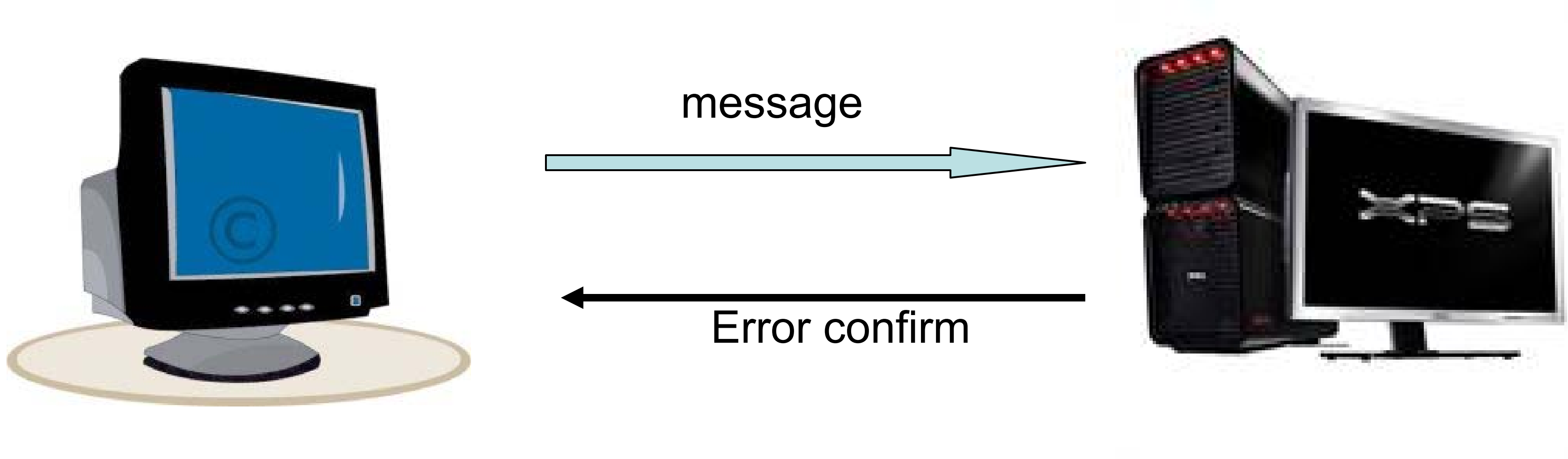
$$P[(k_1,..,k_m)] = \frac{n!}{k_1!k_2!..k_m!} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$$

Geometric probability law

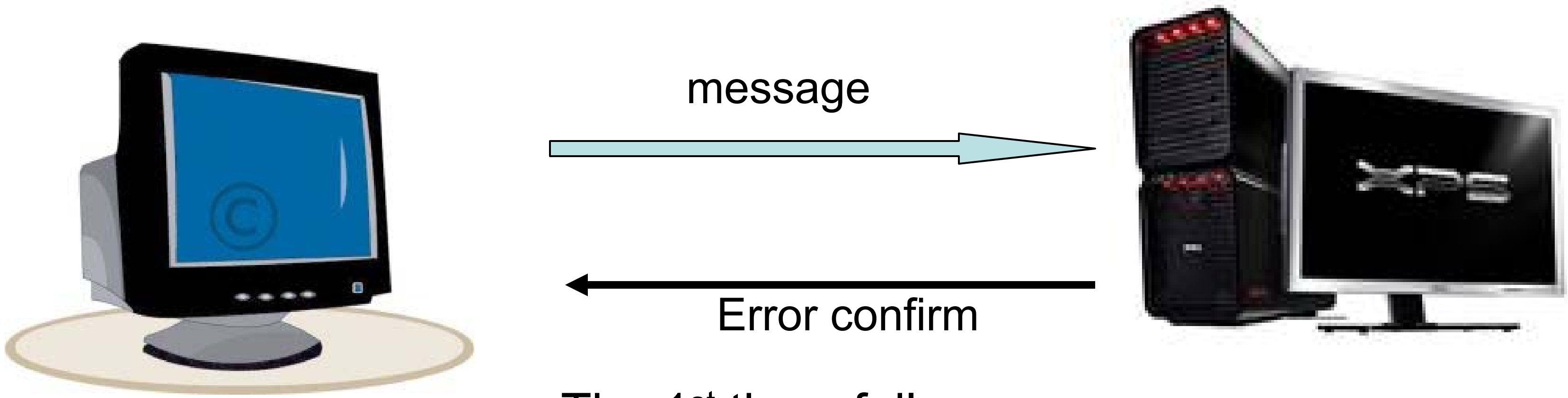
- Bernoulli trial, A is success, A^c is failure. $P[A]=p$, $P[A^c]=1-p$;
- Repeat the trial until the occurrence of the first success (A). Outcome, m, is the # of trails carried out till success.

- $P[m]=P[A^c, A^c,\ldots A^c ,A]=(1-p)^{m-1}p$;
  $m=1,2,\ldots$

Example 3



- The probability that a message needs to be transmitted more than twice?
- Given the probability of a message transmission error is q=0.1.



- The 1st time: fails
- The 2nd time: fails again
- .
- .

Let m be the # of failure before succeeding

P[m>2]

= P[m=3]+P[m=4]+…+P[m=∞]

$=qq(1\text{-}q)+qqq(1\text{-}q)+q^{m\text{-}1}(1\text{-}q)+\ldots$

$$=(1\text{-}q)\sum_{m=2}^{\infty} q^{m} = (1-q)\frac{q^2}{1-q} = q^2$$

- For Bernoulli trial, the probability that more than K trials are needed before succeeding is

$$P[K]=q^k$$

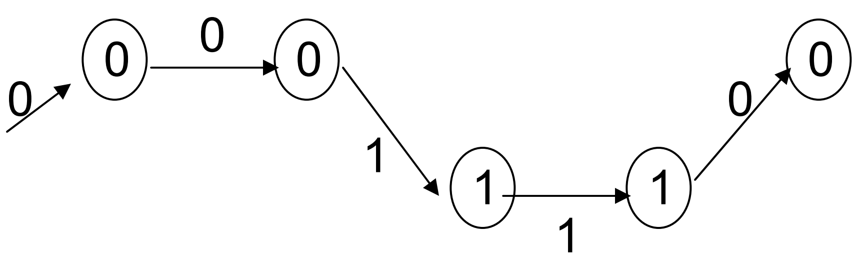
Where q is the failure rate.

Sequence of dependence experiments

- The outcome of a given experiment determines which sub-experiment is performance next, or more generally, influences the outcome of next experiment.

Example 4

Sequence of 1s and 0s : EX: 00110



- Probability of a certain sequence of outcome

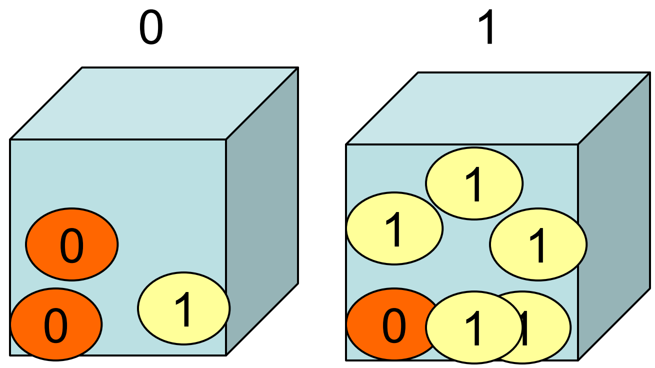
$$P[\ \{S_0\} \cap \{S_1\} \cap \{S_2\}\ ]$$

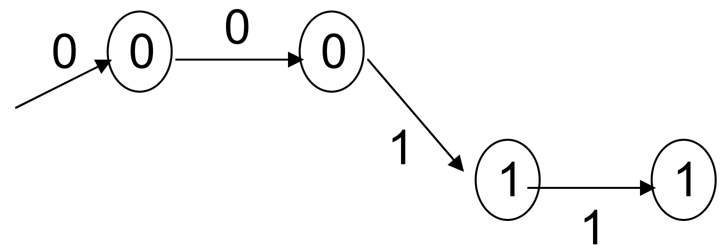
$$= \underbrace{P[\{S_0\}]}\cdot \underbrace{P[\{S_1\} \mid \{S_0\}]}\cdot \underbrace{P[\{S_2\} \mid \{S_0\} \cap \{S_1\}]}$$

Only recent outcome determines sub-experiment

$$= P[\{S_0\}]\cdot P[\{S_1\} \mid \{S_0\}]\cdot P[\{S_2\} \mid \{S_1\}]$$

Markov Chains





P[0 0 1 1]=P[0] P[0|0] P[1|0] P[1|1]

P[0]=1/2

P[0|0]= 2/3

P[1|0]=1/3

P[1|1]=5/6

P[0,0,1,1]=1/2 * 2/3 *1/3 * 5/6 =5/54