ECE 340

Probabilistic methods in engineering

Lecture 4

Jean Liu
Email: Jliu@mrn.org

Reading

- This class: Section 2.5
- Next class: Section 2.6

outline

- Independence
  - Definition derived from conditional probability
  - General definition for two events
  - General definition for three events
  - General definition for N events
- Examples

Independence of events

- If the occurrence of an event A doesn't alter the probability of the occurrence of other event B, then, event A is independent of event B.

$$P[B] = P[B \mid A] = \frac{P[A \cap B]}{P[A]}$$

$$P[A \cap B] = P[A]P[B]$$

- P[A|B] = ?

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

Note: $\mathrm{P[A]} \neq 0$ $\mathrm{P[B]} \neq 0$

Example1

- A ball is selected from an urn including

A: Black balls { (1,b), (2,b) } ½

B: Even-numbered balls {(2,b), (4,w) } ½

C: Greater than 2 balls { {3,w,), (4,w)} ½

Are events A and B independent?
Are events A and C independent?

$$P[A \cap B] = ?$$

$$P[A \cap B] = \{(2, b)\} = \frac{1}{4}$$

$$P[A]P[b] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P[A | B] = ?$$

$$P[A| B] = \frac{P[A \cap B]}{P[B]} = \frac{P\{(2,b)\}}{P\{(2,b),(4,w)\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P[A] = P\{(1,b)(2,b)\} = \frac{1}{2}$$

Events A and B are independent

Events A and C ?

$P[A \cap C] = ?$

$P[A]P[C] = ?$

$P[A | C] = ?$

$P[A] = ?$

$P[A \cap C] = 0$

$P[A]P[C] = 1/4$

$P[A | C] = 0$

$P[A] = 1/2$

Summary for two events

- Events A and B are independent

$$P[A \cap B] = P[A] \cdot P[B]$$

$$p[A] \neq 0; \; p[B] \neq 0$$

Three events

- Conditions for events A, B and C be independent

- 1)pairwise independent

$P[A \cap B]=P[A]P[B];P[A \cap C]=P[A]P[C];P[B \cap C]=P[B]P[C]$

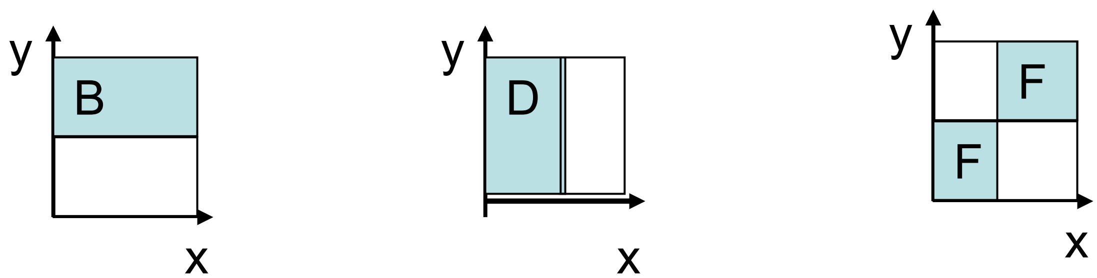
- 2)any joint occurrence of two events should not affect the third one.

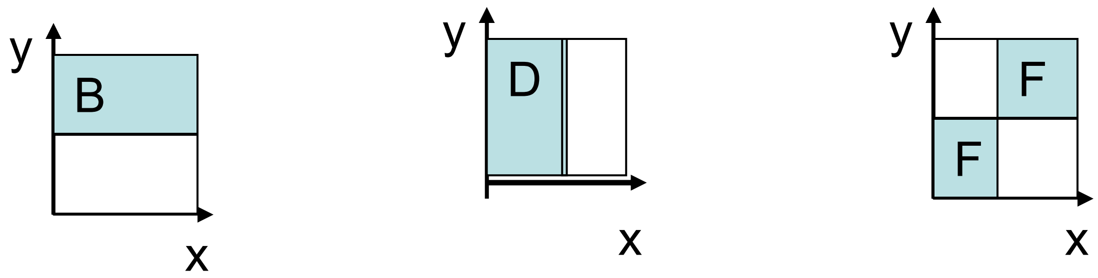
$$P[A|(B \cap C)]=P[A]$$

Example 2

- Two numbers are selected from the unit interval

B={ y>1/2 }   ;   D={ x<1/2 }
F={ x<1/2 and y<1/2} U { x>1/2 amd y>1/2 }





- P[B∩D]=1/4=P[B]P[D]
- P[B∩F]=1/4=P[B]P[F]
- P[D∩F]=1/4=P[D]P[F]

- P[B∩(D∩F)]=0 ≠ P[B] P[D∩F]=1/8

Pair wise independence **does not** imply independence

Independence of three events

$$P[A|(B\cap C)] = P[A]$$

$$P[A|(B\cap C)] = \frac{P[A\cap B\cap C]}{P[B\cap C]} = P[A]$$

$$P[A\cap B\cap C] = P[A]P[B\cap C] = P[A]P[B]P[C]$$

General definition:

$$P[A\cap B\cap C] = P[A]P[B]P[C]$$

Independence of n events

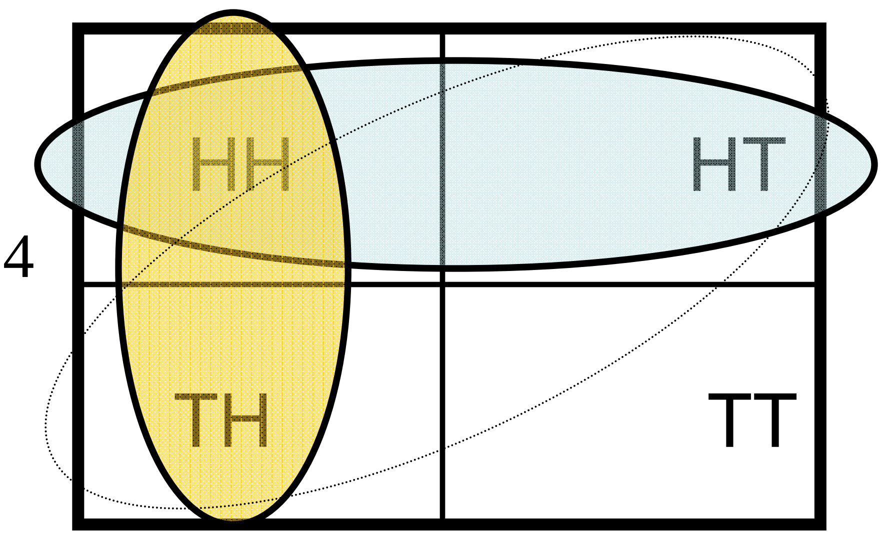
- *General definition:*

- $P[A_{i1} \cap A_{i2} \cap \cdots A_{in}] = P[A_{i1}]P[A_{i2}] \cdots P[A_{in}]$

Independence vs. pare wise independence

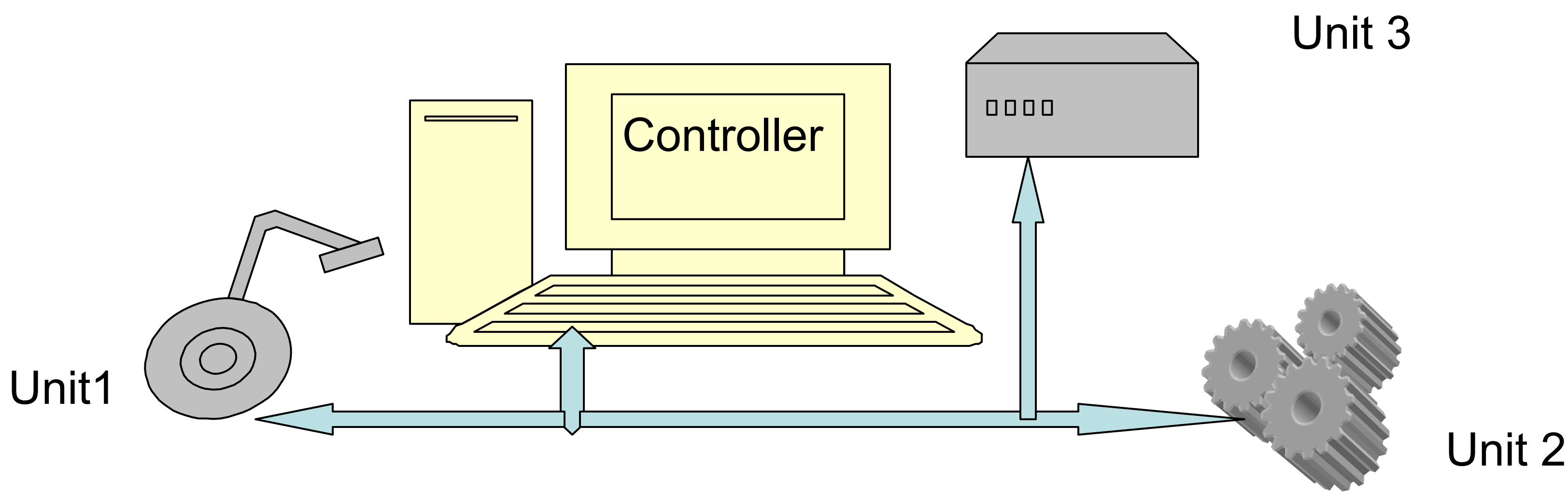
- Two independent fair (p=1/2) coin tosses
- A: First toss is H
- B: Second toss is H.
- C: The two outcome are different

$$P[A] = P[B] = P[C] = 1/2$$
$$P[A \cap B] = P[A \cap C] = p[B \cap C] = 1/4$$
$$P[A | (B \cap C)] = 0$$



Example 3

- System reliability



The system is ‘UP’ if the controller and at least two peripheral units are functioning.

What is the probability that the system is up,
assuming all components fail independently?

- Define “the system is up”

  A: Controller is working

  F: at least two units are functioning.

  B1: Unit 1 is functioning

  B2: Unit 2 is functioning

  B3: Unit 3 is functioning

  $\{B1 \cap B2 \cap B3^C\}$, $\{B1 \cap B3 \cap B2^C\}$,$\{B2 \cap B3 \cap B1^C\}$

  $\{B1 \cap B2 \cap B3\}$

- System up : $\{ A \cap F\}$; $P[A \cap F] =?$

- $P[A \cap F] = P[A] P[F]$

Given:

1) each peripheral unit fails with probability of α, so $P[B_i]=1-\alpha$; $P[B_i^C]= \alpha$;
2) The controller fails with probability of ρ, so $P[A]=1- \rho$;

$P[F]$

$= P[\ \{B1 \cap B2 \cap B3^C\} \cup \{B1 \cap B3 \cap B2^C\} \cup$

$\{B2 \cap B3 \cap B1^C\} \cup \{B1 \cap B2 \cap B3\}\ ]$

$= P[\ \{B1 \cap B2 \cap B3^C\}\ ] + P[\{B1 \cap B3 \cap B2^C\}\ ] +$

$P[\ \{B2 \cap B3 \cap B1^C\}\ ] + P[\{B1 \cap B2 \cap B3\}\ ]$

$= P[B1]\ P[B2]\ P[B3^C] + P[B1]\ P[B3]\ P[B2^C] +$

$P[B2]\ P[B3]\ P[B1^C] + P[B1]P[B2]P[B3]$

$= 3(1-\alpha)^2\alpha + (1-\alpha)^3$

- $P[A \cap F] = P[A] P[F]$
- $=(1-p) \{3(1-\alpha)^2\alpha + (1-\alpha)^3\}$

Assume: $\alpha = 10\%$, $\rho = 20\%$

$P[A \cap F] = (0.8) * \{3 * (0.081) + 0.729\}$

$= 0.8 * 0.972 = 0.78$

Assume: $\alpha = 5\%$, $\rho = 20\%$

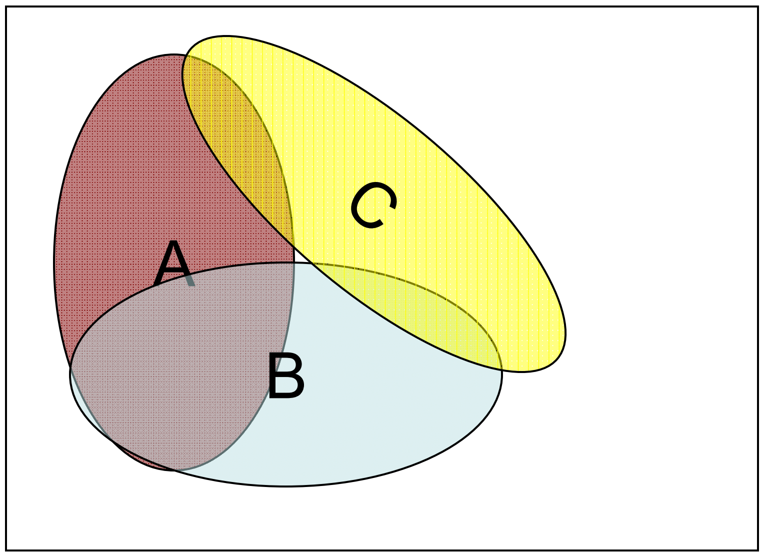
$P[A \cap F] = 0.8 * 0.99 = 0.79$

Assume: $\alpha = 10\%$, $\rho = 10\%$

$P[A \cap F] = 0.9 * 0.972 = 0.87$

Conditioning may affect independence

- Assume A and B are independence



- If C occurred, are A and B independent?

The King’s sibling

- The king comes from a family of two children.
- What is the probability that the king’s sibling is female?

- Homework :
- problems 2.16, 2.62, 2.80, 2.85, 2.86, 2.95

- Due date: Monday, Feb. 11 at the beginning of class.