ECE 340
Probabilistic Methods in Engineering
M/W 3-4:15

Lecture 20: Central Limit Theorem

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Quiz

- **Write down (or compute) the mean and variance of the sample mean for iid random variables with mean m_x and variance σ^2**

- **True or False: The Strong Law of Large Numbers refers to convergence in probability**

- **Section 7.3**
  - **Central Limit Theorem**

Statistical inference

– process by which information from samples data is used to draw conclusions about the population from which the sample was selected.

The Central Limit Theorem

- Let X_1, X_2,… X_n be iid random variables with mean μ and variance σ^2. And let:

$$S_n = X_1 + X_2 + \ldots + X_n$$

If:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

then,

$$\lim_{n \to \infty} P[Z_n \le z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

Proof of the Central Limit Theorem

First, it is important to note that Z_n is a zero-mean unit-variance random variable

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

(Can you remember what are the mean and variance of S_n ?)

Proof of the Central Limit Theorem

Since Z_n is a zero-mean unit-variance random variable:

therefore, we can proof the theorem by showing that the cdf of Z_n converges to the cdf of a Gaussian RV with zero-mean and unit-variance

(This is known as:

"convergence in distribution"

Section 5.5 discusses the different types of convergence sequences of random variables

)

Proof of the Central Limit Theorem

We will derive the cdf (pdf) for Z_n using the characteristic function

In other words, we will need to show that the characteristic function of Z_n converges to the characteristic function of a zero-mean unit-variance Gaussian random variable

$$\Phi_{Z_n}(\omega) = E\left[e^{j\omega Z_n}\right]$$

Proof of the Central Limit Theorem

- We can express Z_n as:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n}(X_k - \mu)$$

- This leads to: $\Phi_{Z_n}(\omega) = E\left[e^{j\omega Z_n}\right]$

$$\Phi_{Z_n}(\omega) = E\left[\exp\left\{\frac{j\omega}{\sigma\sqrt{n}}\sum_{k=1}^{n}(X_k - \mu)\right\}\right]$$

Proof of the Central Limit Theorem

$$\Phi_{Z_n}(\omega) = E\left[\exp\left\{\frac{j\omega}{\sigma\sqrt{n}}\sum_{k=1}^{n}(X_k - \mu)\right\}\right]$$

This can be expressed as:

$$\Phi_{Z_n}(\omega) = E\left[\prod_{k=1}^{n} e^{j\omega(X_k - \mu)/\sigma\sqrt{n}}\right]$$

Proof of the Central Limit Theorem

This expression:

$$\Phi_{Z_n}(\omega) = E\left[\prod_{k=1}^{n} e^{j\omega(X_k-\mu)/\sigma\sqrt{n}}\right]$$

can be simplified to the following:

$$\Phi_{Z_n}(\omega) = \prod_{k=1}^{n} E\left[e^{j\omega(X_k-\mu)/\sigma\sqrt{n}}\right]$$

(This can be done since X_1, X_2 ... are independent)

Proof of the Central Limit Theorem

$$\Phi_{Z_n}(\omega) = \prod_{k=1}^{n} E\left[e^{j\omega(X_k-\mu)/\sigma\sqrt{n}}\right]$$

Since X_1, X_2 … are identically distributed :

$$\Phi_{Z_n}(\omega) = \left\{E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}}\right]\right\}^n$$

Proof of the Central Limit Theorem

$$\Phi_{Z_n}(\omega) = \left\{ E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}} \right] \right\}^n$$

Now let's focus on the term:

$$E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}} \right]$$

Proof of the Central Limit Theorem

$$E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}}\right]$$

This term be be expanded:

$$= E\left[1 + \frac{j\omega}{\sigma\sqrt{n}}(X-\mu) + \frac{(j\omega)^2}{2!n\sigma^2}(X-\mu)^2 + R(\omega)\right]$$

Proof of the Central Limit Theorem

$$E\left[1+\frac{j\omega}{\sigma\sqrt{n}}(X-\mu)+\frac{(j\omega)^2}{2!n\sigma^2}(X-\mu)^2+R(\omega)\right]$$

$$= 1+\frac{j\omega}{\sigma\sqrt{n}}E\left[(X-\mu)\right]+\frac{(j\omega)^2}{2!n\sigma^2}E\left[(X-\mu)^2\right]+E\left[R(\omega)\right]$$

Proof of the Central Limit Theorem

$$1+\frac{j\omega}{\sigma\sqrt{n}}E\left[(X-\mu)\right]+\frac{(j\omega)^2}{2!n\sigma^2}E\left[(X-\mu)^2\right]+E\left[R(\omega)\right]$$

Now, since:

$$E\left[(X-\mu)\right]=0$$

$$E\left[(X-\mu)^2\right]=\sigma^2$$

$$E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}}\right]=1-\frac{\omega^2}{2n}+E\left[R(\omega)\right]$$

Proof of the Central Limit Theorem

$$E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}}\right] = 1 - \frac{\omega^2}{2n} + E\left[R(\omega)\right]$$

We can neglect the term E[R(w)] as n gets large. Also, remember that:

$$\Phi_{Z_n}(\omega) = \left\{E\left[e^{j\omega(X-\mu)/\sigma\sqrt{n}}\right]\right\}^n$$

Proof of the Central Limit Theorem

Therefore as n gets large, we can express the characteristic function:

$$\Phi_{Z_n}(\omega) = \left\{1 - \frac{\omega^2}{2n}\right\}^n$$

Now as n goes to infinity:

$$\lim_{n\to\infty}\left\{1 - \frac{\omega^2}{2n}\right\}^n = e^{-\omega^2/2}$$

Proof of the Central Limit Theorem

Therefore,:

$$\lim_{n\to\infty} \Phi_{Z_n}(\omega) = e^{-\omega^2/2}$$

is the characteristic function of a Gaussian RV with zero-mean and unit-variance

Recall that for a Gaussian random variable X with mean (m) and variance (σ^2), its characteristic function is:

$$\Phi_X(\omega) = e^{j\omega m-\sigma^2\omega^2/2}$$

Proof of the Central Limit Theorem

Also, since the characteristic function and the pdf (cdf) represent a unique pair, then:

$$\lim_{n \to \infty} \Phi_{Z_n}(\omega) = e^{-\omega^2/2}$$

implies,

$$\lim_{n \to \infty} F_{Z_n}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

Proof of the Central Limit Theorem

$$\lim_{n \to \infty} F_{Z_n}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

The above is equivalent to:

$$\lim_{n \to \infty} P[Z_n \leq z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

Central limit theorem

If $\bar{X}$ is the mean of a random sample $X_1, \ldots, X_n$, of size n from a distribution with finite mean μ and finite positive variance σ^2, then the distribution of: $W = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{T_o - n\mu}{\sqrt{n}\sigma}$ is N(0,1) as n $\rightarrow \infty$.

- Important points to notice:
  - When n is "sufficiently large" (n>30), a practical use of the CLT is :

    $$P(W \le w) \approx \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)$$

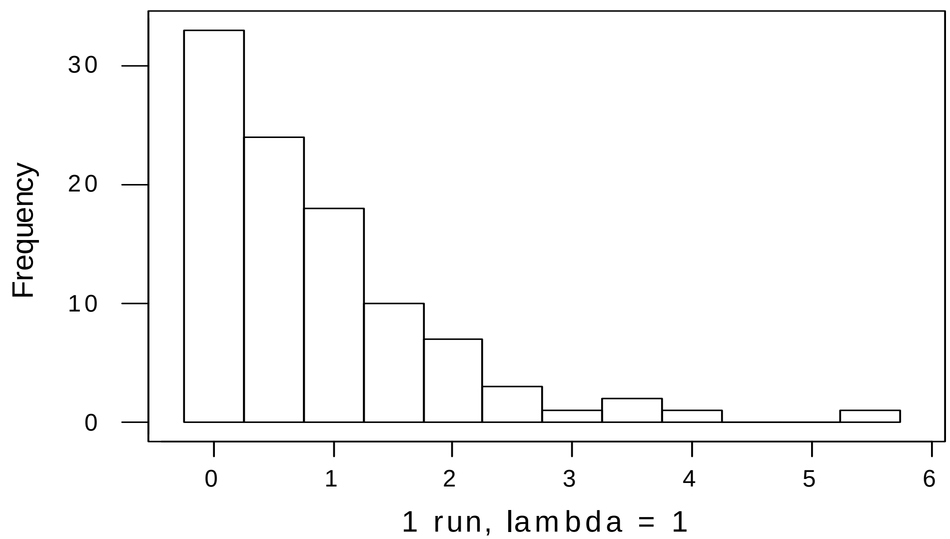
  - The theorem holds for any distribution with finite mean and variance.

Central limit theorem

- What it all means:
  - If n is "large: and we wish to calculate say $P\left(a \leq \overline{X} \leq b\right)$ or P(a≤ T_o ≤ b), we need only "pretend" that $\overline{X}$ or T_o is normal, standardize it, and determine the probabilities from the normal table. The resulting theorem states that it will be approximately correct.

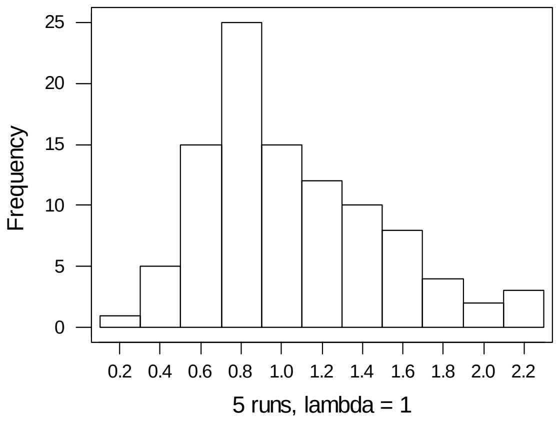
Central Limit Theorem

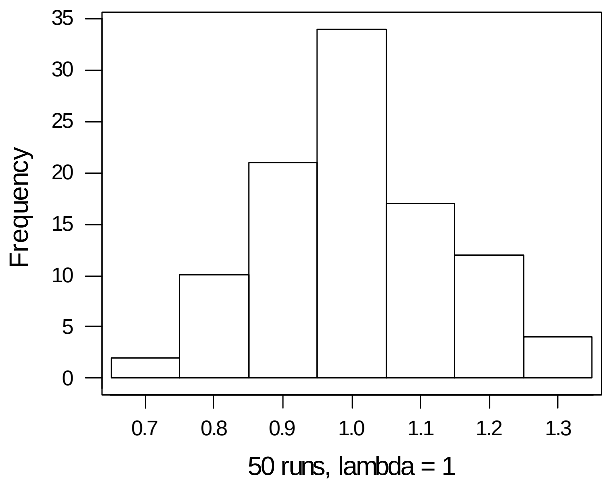
- **Using the exponential distribution and random number generator, it is possible to plot the resulting frequency distributions of data. Notice the trend towards normality.**



Central Limit Theorem

- **Continuing,**





Example: Packet over wireless

- An Internet Protocol (IP) packet consists of n bits, (where n can be larger than 1000). An IP packet is transmitted over a wireless link with a bit error probability p.

  Approximate the probability:

  P[k error bits in a packet]

  using the Central Limit Theorem

Example: Packet over wireless

- Let S_n be the random variable representing the number of bit errors in n bits (i.e., in a packet). Therefore, the probability:

  $$P[\text{k error bits in a packet}] = P[S_n=k]$$

  S_n can be modeled as a binomial random variable with a parameter p (i.e., the probability that the event of interest - bit error event - occurs with a probability p)

Example: Packet over wireless

- S_n is the sum of iid Bernoulli random variables:

$$S_n = I_1 + I_2 + \ldots + I_n$$

- Therefore, as n gets large, the probability cumulative distribution function of S_n converges toward a Gaussian cdf with:

$$E[S_n] = np \quad \text{and} \quad \sigma^2 = np(1-p)$$

Example: Packet over wireless

$P[S_n=k] \rightarrow N\ (\ np\ ;\ np(1-p)\ )$

$$P\left[S_n = k\right] \cong \frac{1}{\sqrt{2\pi np(1-p)}} e^{-(k-np)^2/2np(1-p)}$$

This is a general approximation of the pmf of a binomial random variable with parameters (p) and (n) using a Gaussian pdf

T-Distribution

t-distribution or student's t-distribution

Using S for σ in computing standardized z-values to look up on the normal table is not trustworthy for small sample sizes (n<30).

Why? Because the CLT only applies to "large" samples.

As a result, when n is small and/or σ is unknown, you must use an alternate distribution, the t-distribution.

Theorem – When $\overline{X}$ is the mean of a random sample of size n from a normal distribution with mean μ, the random variable

$T = \frac{\overline{X} - \mu}{s/\sqrt{n}}$ ~ t-distribution with n-1degrees of freedom.

Note: A t-distribution has 1 parameter called the degrees of freedom, ν. Possible values of ν are 1, 2, …. Each different value of ν corresponds to a different t-distribution. ⇒ df = n-1.

T-Distribution

Properties of the t-distribution:

each t_ν curve is bell-shaped and centered at zero.

as ν increases, the spread of the corresponding t_ν curve decreases.

each t_ν curve is more spread out then the standard normal(z) curve.

as $\nu \rightarrow \infty$, the t_ν curve approaches the standard normal curve.

Let $t\alpha,\nu$ = the point on the t-distribution with νdf, such that the area to the right is α.