ECE 340
Probabilistic Methods in Engineering
M/W 3-4:15

Lecture 1_9: Review

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Review

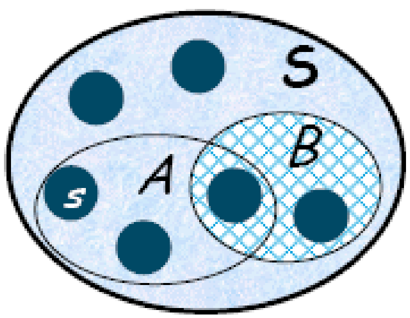
- Section 2.1
  - Examples of experiments
  - Sample Space & Examples
    - Discrete
    - Continuous
  - Events & Examples
    - Certain event
    - Null event
  - Set Operations
    - Union
    - Intersection
    - Complement

Outcomes; events; sample space

- **An event A is a set of outcomes:**

  A = { s : such that s is an even number }

"outcome" $\in$ Event $\subset$ Sample Space



$s \in A \subset S$

Properties of Probability

a. **Many times,** $P(A) = \frac{N(A)}{N(S)}$ $\frac{\text{\# of ways A can occur}}{\text{\# of ways S can occur}}$

Relative frequency of percentage

b. **Properties**

i. **For an event A,** **P(A) = 1 – P(A)’**
P(A’) = 1 – P(A)

ii. **If 2 events, A & B are mutually exclusive, then $P(A \cap B) = 0$.**

iii. **For 2 events A & B,**
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

iv. **For 3 events A, B, C**
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Probability Theory

Some Probability Laws

Commutative Laws: $A \cup B = B \cup A$

$A \cap B = B \cap A$

Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgin's Laws: $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

Review

- Section 2.3
  - Counting
    - Sampling w/ & w/o replacement
    - Permutations of n objects

Enumeration or Counting Technologies

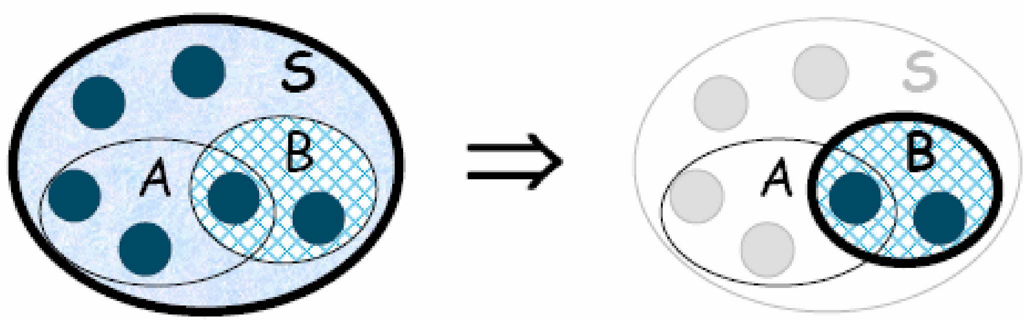
Summary

	With Replacement	Without Replacement
Order relevant (ab $\neq$ ba)	$n_1 \times n_2 \times \ldots n_r$	$P_r^n = \frac{n!}{(n-r)!}$
Order irrelevant (ab = ba)	$\frac{(n-1+r)!}{(n-1)!r!}$	$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

1,1	**2, 1**	**1, 2**	**2, 1**
1,2	**2, 2**		

Conditional Probabilities

- Given that an event B has occurred, what is the probability of A
- Given that B has occurred, reduces the sample space: $S \rightarrow B \subset S$



Review

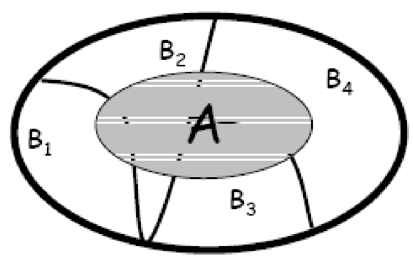
- Section 2.4
  - Conditional probability
    - Bayes' Rule

Total Probability

- If B_1 , B_2 ,…B_n form a "partition" of S, then for any event A:

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset \ , \ i \neq j$$

$$\Rightarrow \quad A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots (A \cap B_n)$$



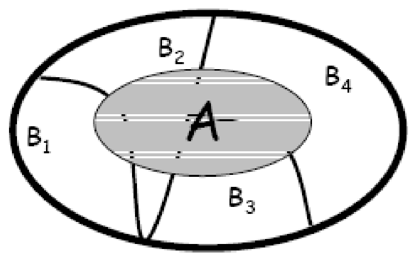
Total Probability

- Since A can be expressed as the union of mutually exclusive events:

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots (A \cap B_n)$$

$\Rightarrow$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \dots P[A \cap B_n]$$

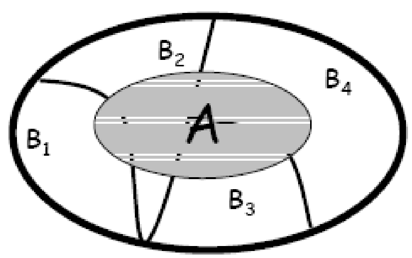


Total Probability

- **The Law of Total Probability:**

If B_1 , B_2 ,... form a "partition" of S, then for any event A:

$$P[A] = P[A/B_1].P[B_1] + P[A/B_2].P[B_2] \ldots$$

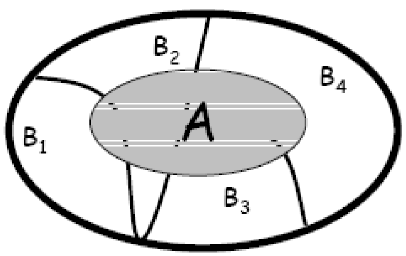


$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots$$

Bay's Rule

- If B_1 , B_2 ,…B_n form a "partition" of S, then for any event A:

$$P[B_j/A] = \frac{P[A/B_j]\cdot P[B_j]}{\sum_{i=1}^{n} P[A/B_i]\cdot P[B_i]}$$



Review

- Section 2.5
  - Definition of independence

Independence of events

- **If the occurrence of an event A doesn't alter the probability of the occurrence of other event B, then, event A is independent of event B.**

$$P[B] = P[B \mid A] = \frac{P[A \cap B]}{P[A]}$$

$$P[A \cap B] = P[A]P[B]$$

Independence of events

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The King's sibling

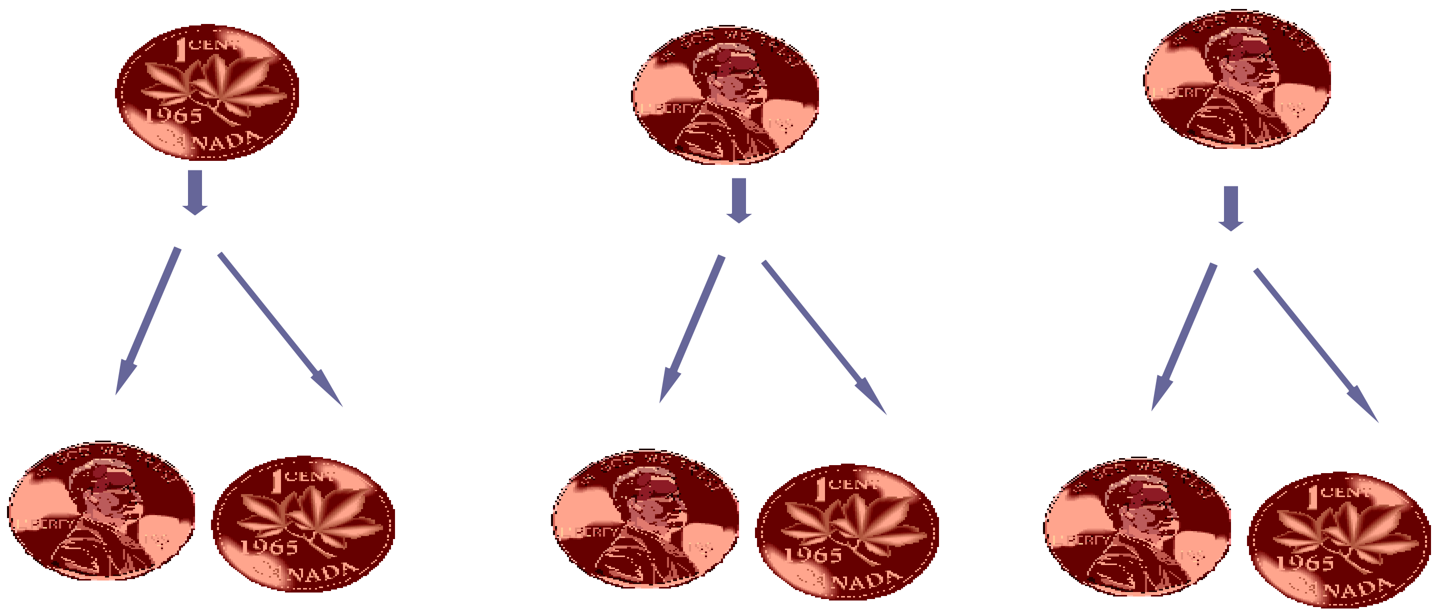
- The king comes from a family of two children.
- What is the probability that the king's sibling is female?

Review

- **Section 2.6**
  - **Sequences of independent experiments**
  - **binomial probability law**
  - **multinomial probability law**
  - **geometric probability law**
  - **sequences of dependent events**

Example1
Sequences of independent experiment

- Flip coin 3 times experiment
- Each sub-experiment is independent. The probability of heads is ρ



- Sample Space:{ {H,T } X {H,T} X {H,T} }

Binomial vs. Multinomial

- Independent
- M=2 (Success/ Failure)
- P[S]=p; P[F]= 1-p
- Repeat N time

- Independent
- M=m (B1, B2, B3…Bm)
- P[Bi]=pi ; (Σpi=1)
- Repeat N time

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for k=0,1,...n

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$P[(k_1, .., k_m)] = \frac{n!}{k_1 ! k_2 !..k_m !} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$$

Sequence of dependence experiments

- The outcome of a given experiment determines which sub-experiment is performance next, or more generally, influences the outcome of next experiment.

- **Probability of a certain sequence of outcome**

$P[\{S_0\} \cap \{S_1\} \cap \{S_2\}]$

$= P[\{S_0\}] \cdot P[\{S_1\} \mid \{S_0\}] \cdot P[\{S_2\} \mid \{S_0\} \cap \{S_1\}]$

Only recent outcome determines sub-experiment

$= P[\{S_0\}] \cdot P[\{S_1\} \mid \{S_0\}] \cdot P[\{S_2\} \mid \{S_1\}]$

Markov Chains

Review

- Section 3.1
  - measurement
  - sample space
- Section 3.2
  - Discrete RV & PMF
  - Examples

Probability mass function

$$(I) \qquad p_X(x) \geq 0 \qquad \text{for all x}$$

$$(II) \qquad \sum_{x \in S_X} p_X(x) = 1$$

$$(III) \qquad P[X \; in \; B] = \sum_{x \in B} p_X(x)$$

Review

- **3.3**
  - **expected value and moments**
  - **expected value of functions of RVs**
  - **Variance of an RV**

Expected value (mean)

- **Definition:**

$$E[X] = \sum_{x \in S_X} x \cdot p_X(x) = \sum_k x_k \cdot p_X(x_k)$$

- **Interpretation**
  - **Center of gravity**
  - **Average in a large number of repetitions (infinite)**

Review

- **Section 3.4**
  - **Conditional PMF**
  - **conditional expected value**

Conditional probability mass function

- **Let X be a discrete random variable with a pmf $p_X(x)$. Let C be an event with P[C]>0.**

The conditional probability mass function of X : $p_X(x|C) = P[X=x \mid C]$

$$= \frac{P[\{X=x\} \cap C]}{P[C]}$$

Conditional expected value

- **E[X|B]**

$$E[X \mid B] = \sum_{x \in S_X} x \cdot p_X(x \mid B)$$

Review

- Section 3.5
  - Important discrete RV's
    - Bernoulli
    - Binomial
    - Geometric
    - Negative binomial
    - Poisson
  - Generation of discrete RV's

Common Families of Discrete Probability Distribution

Bernoulli B(1,p)	$p(x) = f(x) = p^x(1-p)^{1-x}$, $x = 0, 1$ $M(t) = 1 - p + pe^t$ $\mu = p$, $\sigma^2 = p(1-p) = pq$
Binomial B(n, p)	$p(x) = f(x) = \dfrac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, x = 0,1...,n$ $M(t) = (1 - p + pe^t)^n$ $\mu = p$, $\sigma^2 = p(1-p) = pq$
Geometric	$p(x) = f(x) = (1-p)^{x-1}p$, $x = 1, 2, \ldots$ $M(t) = \dfrac{Pe^t}{1-(1-p)e^t}, t < -\ln(1-p)$ $\mu = \dfrac{1}{p}$, $\sigma^2 = \dfrac{1-p}{p^2} = \dfrac{q}{p^2}$
Poisson	$p(x) = f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0,1, \ldots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda$, $\sigma^2 = \lambda$

Review

- Section 4.1-4.3
  - CDF
  - PDF
  - Expected value of X

Continuous Random Variables

i. Let X be a continuous r.v. The probability distribution or probability density function (p.d.f.) of X is a function f(x) [i.e. p(x)] such that for any 2 numbers a and b with a < b

$$P(a \le x \le b) = \int_a^b f(x)dx$$

that is, the probability X takes on a value in the interval [a,b] is the area under the graph ot the density function.

In order that f(x) be a legitimate p.d.f, it must satisfy 2 conditions:

1. f(x) $\geq$ 0, for all x
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Probability Density Function pdf

- Probability that a continuous random variable X takes on values between a and b:

$$P[a \le X \le b] = \int_a^b f_X(x)\,dx$$

Cumulative Distribution Function (cdf)

- cdf $F_X(a)$ of a random variable X is defined as "the probability that X has a value smaller than <u>or</u> equal to a":

$$F_X(a) = P[X \le a] \qquad \forall \ -\infty \le a \le +\infty$$

Expected Value for a Continuous R.V.

a. For the discrete r.v. X, E[X] was obtained by summing xp(x) over possible x values. Here we replace summation by integration and the p.m.f by pdf to get a continuous weighted average.

   i. Def: The expected value or mean value of a continuous r.v. X with pdf f(x) is:

$$\mu_x = E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Moments of random variables

- **Special cases of $E[(X-a)^n]$:**
  - The mean: $E[X] = m$, $a = 0$, $n = 1$
  - The second moment: $E[X^2]$, $a = 0$, $n = 2$
  - The central moments: $E[(X-m)^n]$, $a = m = E[X]$
  - The variance: $E[(X-m)^2]$, $a = m = E[X]$, $n = 2$