ECE 340
Probabilistic Methods in Engineering
M/W 3-4:15

Lecture 15: Function of 2 RV's

Prof. Vince Calhoun

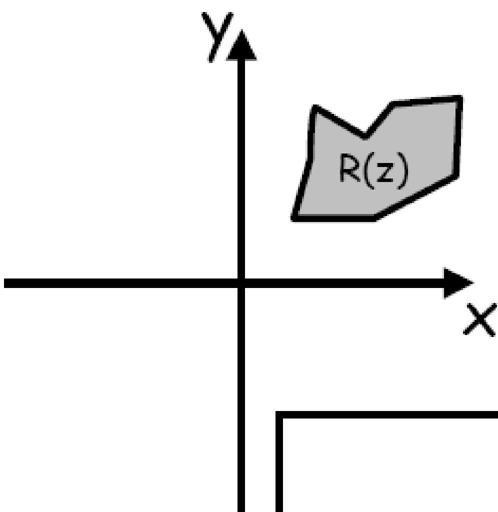
- **Section 5.8-5.9**
  - **Function of two RV's**
  - **Pairs of jointly Gaussian RV's**

Functions of Two Random Variables

- In general, we have to find the pdf and cdf of a random variable $Z=g(X,Y)$
- Start with the event $\{Z \leq z\}$
- This leads to the event $\{g(X,Y) \leq z\}$
- This leads to the event $\{(X,Y) \in R(z)\}$, where $R(z)$ is some 2-D region in the (x,y) plane

Functions of Two Random Variables

- The probability of the event $\{(X,Y) \in R(z)\}$:



$$P[Z \le z] = P[\ \{(X,Y) \in R(z)\}\ ]$$

$$F_Z(z) = \iint_{R(z)} f_{XY}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d\left(\displaystyle\iint_{R(z)} f_{XY}(x,y)\,dx\,dy\right)}{dz}$$

Functions of Two Random Variables

- In summary, for Z=g(X,Y)

$$\{Z \le z\} \Rightarrow \{g(X,Y) \le z\} \Rightarrow \{(X,Y) \in R(z)\}$$

- Examples:

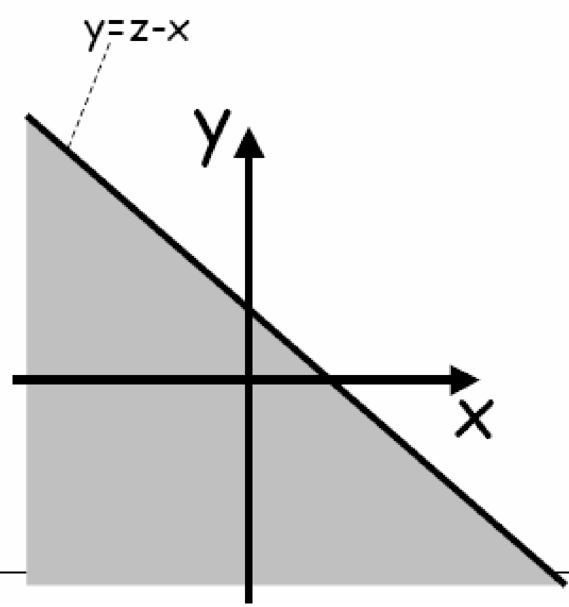
$$Z = X + Y$$

$$Z = X / Y$$

$$Z = \sqrt{X^2 + Y^2}$$

Example I.8

- Let X and Y be two random variables with a joint pdf $f_{XY}(x,y)$. Find the cdf and pdf of the random variable $Z=X+Y$.



$$P[Z \le z] = P[X+Y \le z]$$

$$P[X+Y \le z] =$$

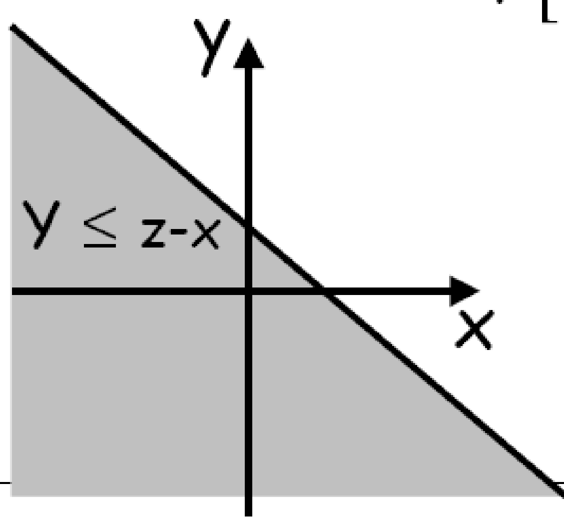
$$P[\{Y \le z-x\} \cap \{-\infty \le X \le +\infty\}]$$

Example I.8

$P[Z \le z] = P[X+Y \le z] =$

$P[\{Y \le z-x\} \cap \{-\infty \le X \le +\infty\}]$

$$P[Z \le z] = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx$$



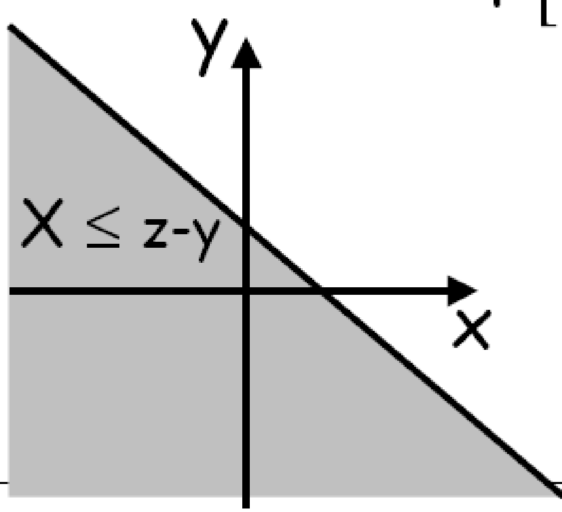
$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx$$

Example I.8

$P[Z \leq z] = P[X+Y \leq z] =$

$P[\{X \leq z\text{-}y\} \cap \{-\infty \leq Y \leq +\infty\}]$

$$P[Z \leq z] = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy$$

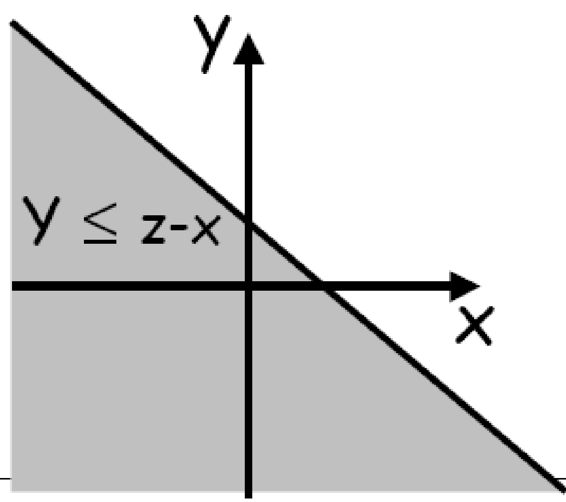


$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy$$

Example I.8

$$F_z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

$$F_z(z) = \int_{-\infty}^{\infty} h(z,x)\,dx$$

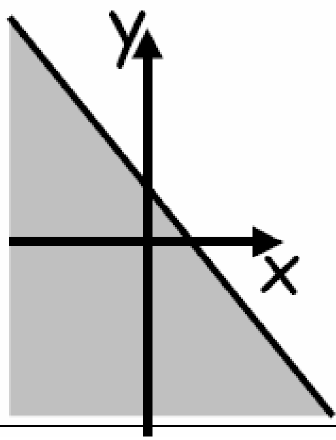


$$h(z,x) = \int_{-\infty}^{z-x} f_{XY}(x,y)\,dy$$

Example I.8

$$F_Z(z) = \int_{-\infty}^{\infty} h(z,x)\, dx \qquad f_Z(z) = \frac{dF_Z(z)}{dz}$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dz}\big(h(z,x)\big)\, dx$$



$$\frac{d}{dz}\big(h(z,x)\big) = \frac{d}{dz}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\right)$$

Example I.8

$$\frac{d}{dz}\big(h(z,x)\big) = \frac{d}{dz}\left(\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right)$$

$$\frac{d}{dz}\big(h(z,x)\big) = f_{XY}(x,z-x)$$

$$f_Z(z) = \frac{dF_z(z)}{dz} = \int_{-\infty}^{\infty} \frac{d}{dz}\big(h(z,x)\big)\,dx$$

$$f_Z(z) = \frac{dF_z(z)}{dz} = \int_{-\infty}^{\infty} f_{XY}(x,z-x)\,dx$$

Example I.8

- Therefore, for Z=X+Y:

$$F_z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dy\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy$$

Example I.8

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx$$

When X and Y are independent:

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

Example I.8

- **When Z=X+Y <u>and</u> X & Y are independent:**

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

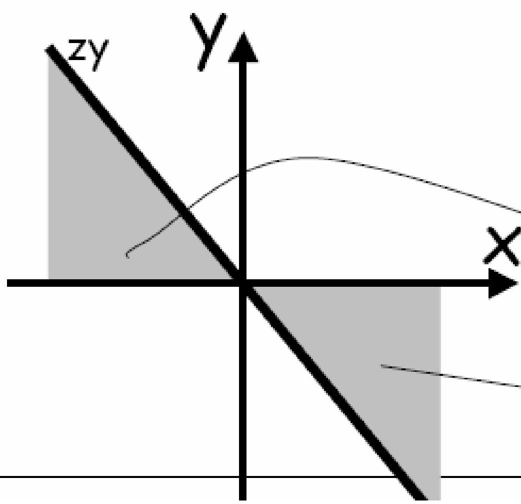
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, dy$$

f_Z is the convolution of f_X and f_Y

Example I.9

- **Let X and Y be two random variables with a joint pdf $f_{XY}(x,y)$. Find the cdf and pdf of the random variable Z=X/Y.**

$$P[Z \le z] = P\left[\frac{X}{Y} \le z\right]$$



$$\{X/Y \le z\} =$$

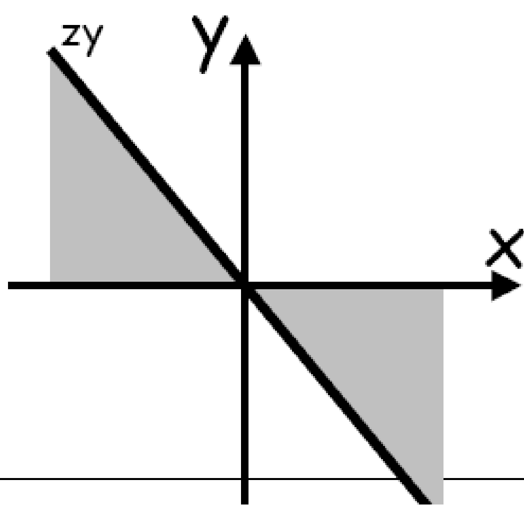
$$\{\ \{X \le z\,y\} \cap \{\ 0 < Y \le +\infty\}\ \}\ \cup$$

$$\{\ \{X \ge z\,y\} \cap \{-\infty \le Y < 0\ \}\ \}$$

Example I.9

$$\Rightarrow P[Z \le z] = P[\{X \le z\,y\} \cap \{Y > 0\}]$$

$$+ P[\{X \ge z\,y\} \cap \{Y < 0\}]$$

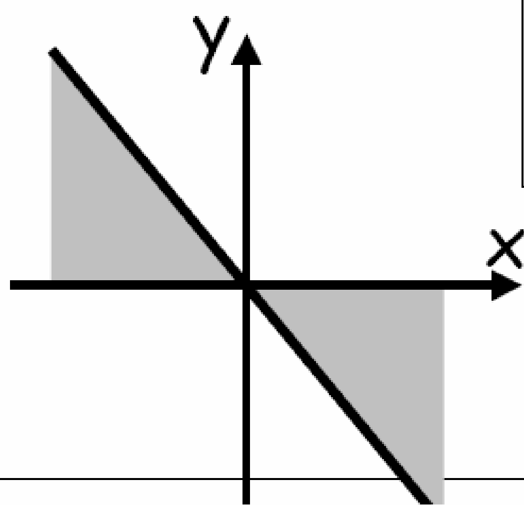


$$F_Z(z) = \int_0^{\infty} \int_{-\infty}^{y\,z} f_{xy}(x,y)\,dx\,dy$$

$$+ \int_{-\infty}^{0} \int_{y\,z}^{\infty} f_{xy}(x,y)\,dx\,dy$$

Example I.9

$$F_Z(z) = \int_0^{\infty} \int_{-\infty}^{yz} f_{xy}(x,y)\,dx\,dy + \int_{-\infty}^{0} \int_{yz}^{\infty} f_{xy}(x,y)\,dx\,dy$$



$$F_Z(z) = \int_0^{\infty} h_1(y,z)\,dy + \int_{-\infty}^{0} h_2(y,z)\,dy$$

Example I.9

$$F_Z(z) = \int_0^{\infty} h_1(y,z)\,dy + \int_{-\infty}^{0} h_2(y,z)\,dy$$

$$f_Z(z) = \frac{dF_z(z)}{dz}$$

$$= \int_0^{\infty} \frac{d}{dz}\left(h_1(y,z)\right) dy + \int_{-\infty}^{0} \frac{d}{dz}\left(h_2(y,z)\right) dy$$

Example I.9

$$\frac{d}{dz}\left(h_1(y,z)\right) = \frac{d}{dz}\left(\int_{-\infty}^{zy} f_{XY}(x,y)\,dx\right)$$

$$\boxed{\frac{d}{dz}\left(h_1(y,z)\right) = y f_{XY}(zy,y)}$$

$$\frac{d}{dz}\left(h_2(y,z)\right) = \frac{d}{dz}\left(\int_{zy}^{\infty} f_{XY}(x,y)\,dx\right)$$

$$\boxed{\frac{d}{dz}\left(h_2(y,z)\right) = -y f_{XY}(zy,y)}$$

Example I.9

$$f_Z(z) = \int_0^{\infty} \frac{d}{dz}\left(h_1(y,z)\right) dy + \int_{-\infty}^{0} \frac{d}{dz}\left(h_2(y,z)\right) dy$$

$$f_Z(z) = \int_0^{\infty} y f_{XY}(zy,y)\, dy - \int_{-\infty}^{0} y f_{XY}(zy,y)\, dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} |\, y \,|\, f_{XY}(zy,y)\, dy$$

Example I.9

- **Therefore, for Z=X/Y:**

$$F_Z(z) = \int_0^{\infty} \int_{-\infty}^{yz} f_{xy}(x,y)\,dx\,dy + \int_{-\infty}^{0} \int_{yz}^{\infty} f_{xy}(x,y)\,dx\,dy$$

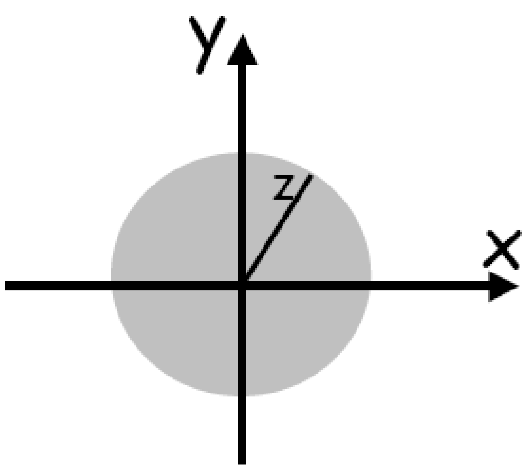
$$f_Z(z) = \int_0^{\infty} y f_{XY}(zy,y)\,dy - \int_{-\infty}^{0} y f_{XY}(zy,y)\,dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_{XY}(zy,y)\,dy$$

Example I.10

- Let X and Y be circularly symmetrical random variables with a joint pdf $f_{XY}(x,y)$. Find the cdf and pdf of:

$$Z = \sqrt{X^2 + Y^2}$$



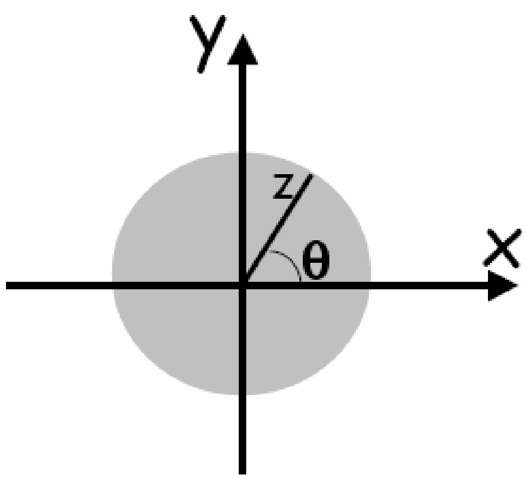
$$P[Z \le z] = P\left[\sqrt{X^2 + Y^2} \le z\right]$$

$$= P\left[X^2 + Y^2 \le z^2\right]$$

Example I.10

- **Since X and Y are circularly symmetrical:**

$f_{XY}(x,y)=g(r)$ **where,** $r = \sqrt{x^2 + y^2}$



$$F_Z(z) = \int_0^z \int_0^{2\pi} g(r)(r d\theta) dr$$

$$F_Z(z) = 2\pi \int_0^z r g(r) dr$$

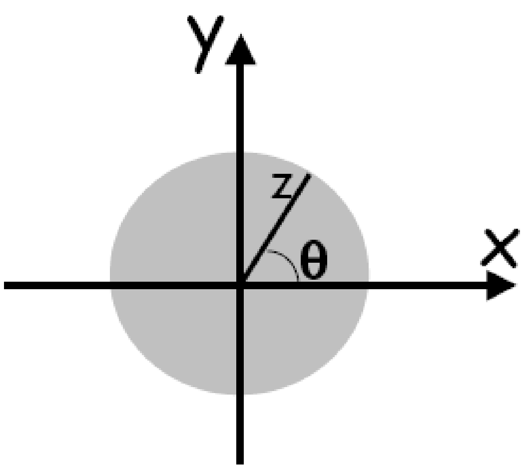
Example I.10

Therefore, for circularly symmetrical X & Y,

the random variable $Z = \sqrt{X^2 + Y^2}$

$$F_Z(z) = 2\pi \int_0^z r g(r) dr$$

$$f_Z(z) = \frac{dF_z(z)}{dz}$$



$$f_Z(z) = 2\pi z g(z)$$

Example I.11

Let X & Y be jointly Gaussian (normal) and circularly symmetrical RVs. Find the pdf and cdf of the random variable: $Z = \sqrt{X^2 + Y^2}$

<u>Solution</u>

Since X & Y are normal and C.S. , then they must have:

- Zero means
- A zero correlation coefficient (for normal RVs this implies independence)
- Equal variances

Example I.11

Therefore

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\left(x^2 + y^2\right)/2\sigma^2}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} = g(r)$$

Now using: $$\boxed{f_Z(z) = 2\pi z g(z)}$$

Example I.11

$$f_Z(z) = 2\pi z\left(\frac{1}{2\pi\sigma^2}e^{-z^2/2\sigma^2}\right)$$

$$\boxed{f_Z(z) = \frac{z}{\sigma^2}e^{-z^2/2\sigma^2}} \qquad z \geq 0$$

Hence, z has a Rayleigh density function

Functions of Random Variables

Therefore, when X & Y are jointly Gaussian (normal) and circularly symmetrical RVs, then the random variable:

$$Z = \sqrt{X^2 + y^2}$$

is a Rayleigh random variable