ECE 340
Probabilistic Methods in Engineering

M/W 3-4:15

Lecture 11: Chebychev, Markov, Transform

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Quiz

- **Write the pdf of a Gaussian random variable**
- **What’s the difference between a normal random variable and a standard normal random variable**

Reading

- This class: Section 4.6-4.7
- Next class: Section 5.1-5.2

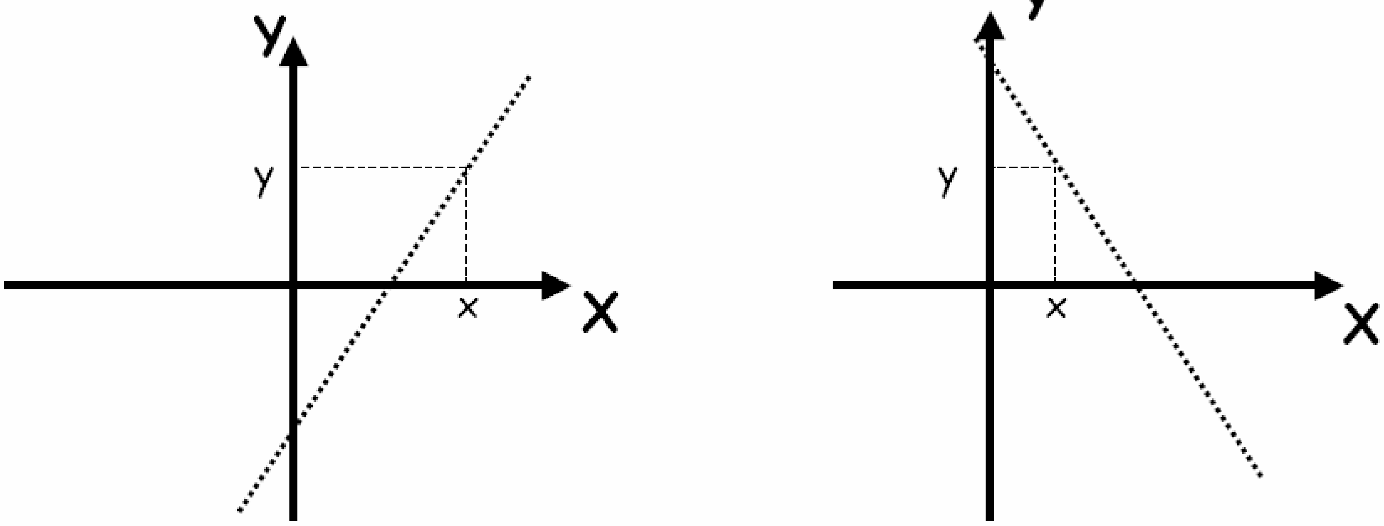
Functions of RVs

Functions of Random Variables

- **Functions of random variables are also random variables**
- **The pdf of the random variable $Y=g(X)$ depends on:**
    - The function $g(x)$
    - The pdf of the random variable X: $f_X(x)$

Example I.4

- Let Y=aX+b, where a and b are some non-zero constants and X is a RV with cdf $F_X(x)$. Find the cdf and pdf for X.



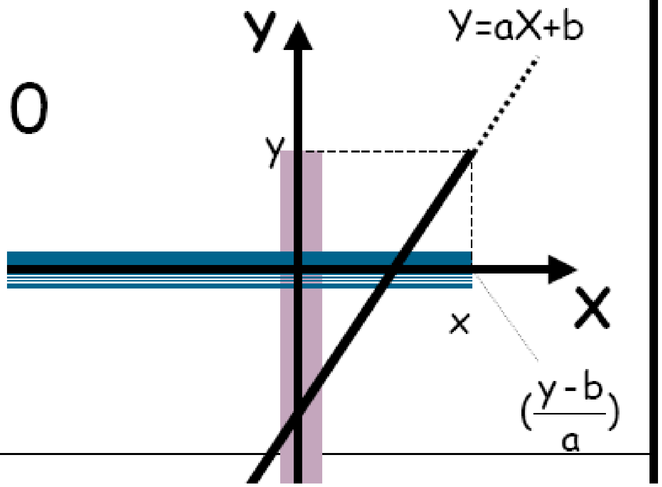
Example I.4

- $F_Y(y) = P[Y \leq y]$

$$P[Y \leq y] = P[(aX + b) \leq y]$$

$$P[Y \leq y] = P[X \leq (\frac{y-b}{a})] \quad \text{if } a > 0$$

$$P[Y \leq y] = F_X(\frac{y-b}{a}) \quad \text{if } a > 0$$



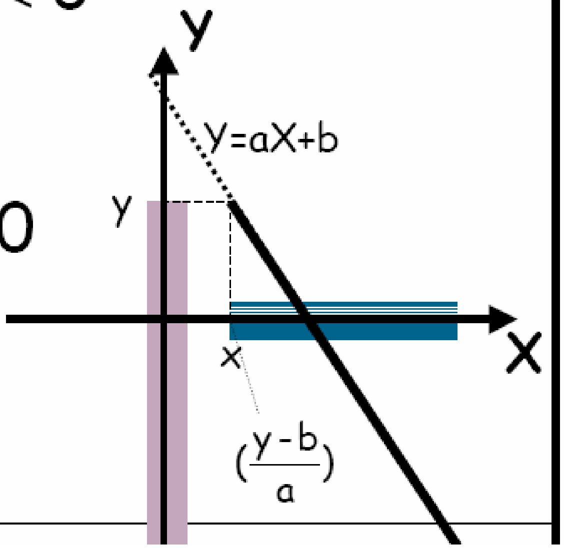
Example I.4

- $F_Y(y) = P[Y \leq y]$

$$P[Y \leq y] = P[(aX + b) \leq y]$$

$$P[Y \leq y] = P[X \geq (\frac{y - b}{a})] \quad \text{if } a < 0$$

$$P[Y \leq y] = 1 - F_X(\frac{y - b}{a}) \quad \text{if } a < 0$$



Example I.4

- Therefore, for Y=aX+b

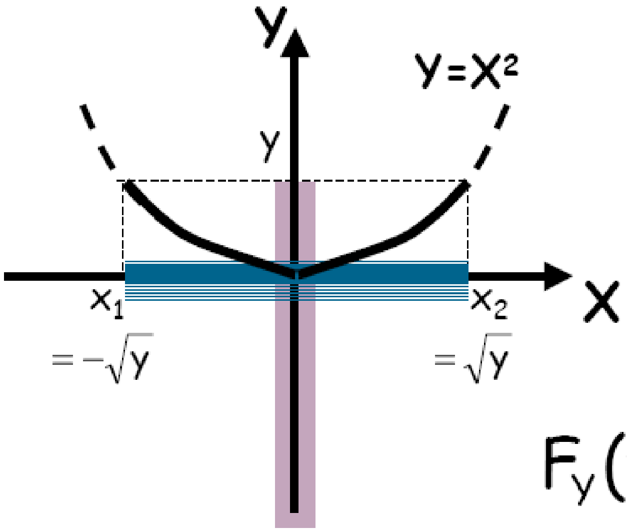
$$P[Y \le y] = F_Y(y) = F_X(\frac{y-b}{a}) \quad \text{if } a > 0$$

$$P[Y \le y] = F_Y(y) = 1 - F_X(\frac{y-b}{a}) \quad \text{if } a < 0$$

- Derive $f_Y(y)$ is terms of f_X

Example I.5

- Let $Y=X^2$, where X is a RV with cdf F_X. Derive F_Y in terms of F_X .



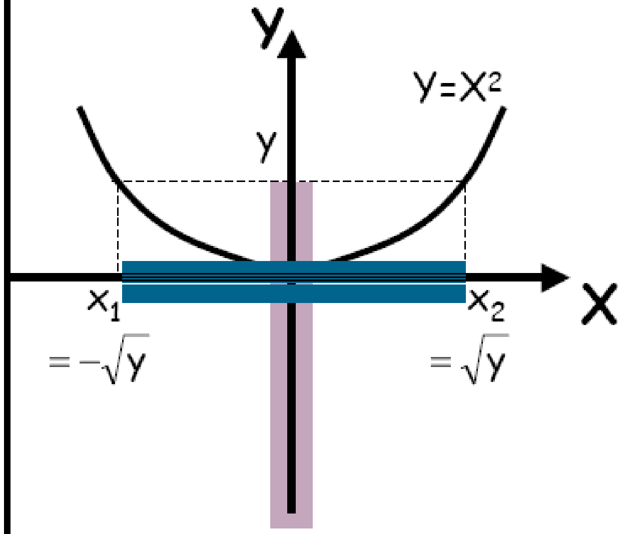
$$F_Y(y) = P[Y \le y]$$

$$Y \le y \quad \Leftrightarrow \quad x_1 \le X \le x_2$$

$$F_Y(y) = P[Y \le y] = P[x_1 \le X \le x_2]$$

Example I.5

$$F_Y(y) = P[Y \le y] = P[x_1 \le X \le x_2]$$



Remember, in general:

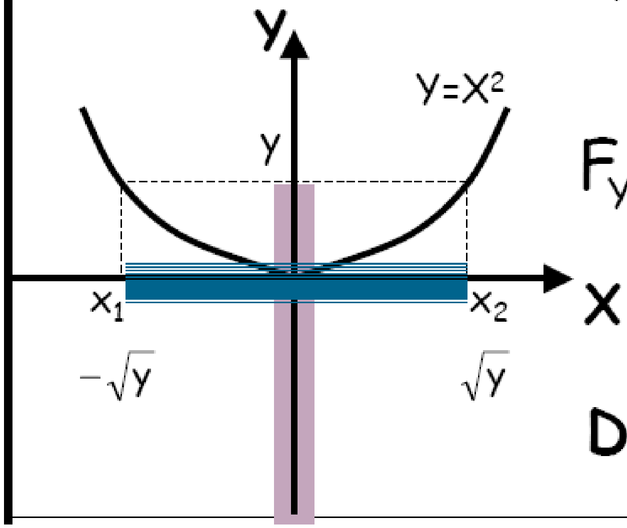
$$P[x_1 \le X \le x_2] =$$

$$F_X(x_2) - F_X(x_1^-)$$

Example I.5

For a RV with a continuous cdf:

$$P[x_1 \le X \le x_2] = F_X(x_2) - F_X(x_1^-) = F_X(x_2) - F_X(x_1)$$



$$F_Y(y) = F_X(x_2) - F_X(x_1)$$

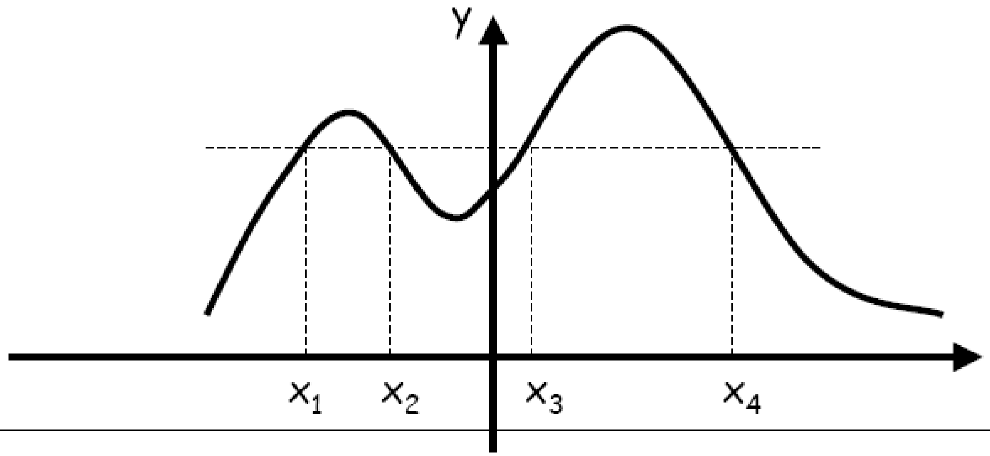
$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad \forall\, y \ge 0$$

Derive $f_Y(y)$ is terms of f_X

Fundamental Theorem

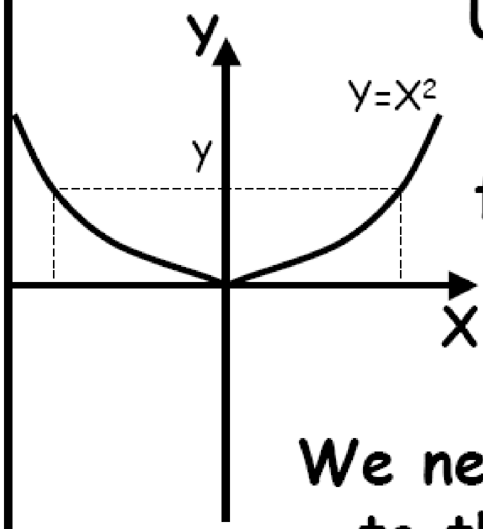
- If x_1 , x_2 , x_3 , …. x_n , …. are solutions to the equation $y=g(x)$, then the pdf of the RV Y can be expressed as:

$$f_Y(y) = \frac{f_X(x_1)}{|dg(x_1)/dx|} + \frac{f_X(x_2)}{|dg(x_2)/dx|} + \cdots$$



Example I.6

- Let $Y=X^2$, where X is a RV with pdf f_X. Derive f_Y in terms of f_X .



Using the fundamental theorem,

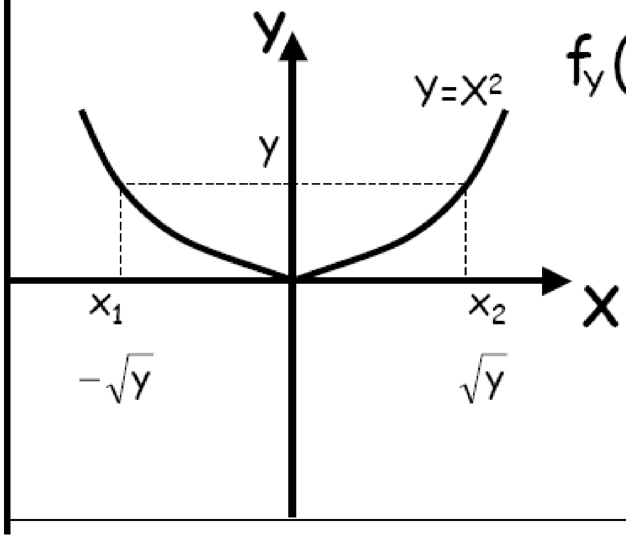
$$f_Y(y) = \frac{f_X(x_1)}{|dg(x_1)/dx|} + \frac{f_X(x_2)}{|dg(x_2)/dx|} + \cdots$$

We need to find the solutions x_1, x_2, x_3 ,… to the equation $y=g(x)=x^2$

Example I.6

The equation $y=g(x)=x^2$, has two solutions:

$$x_1 = -\sqrt{y} \quad \text{and} \quad x_2 = \sqrt{y} \quad \forall\, y \geq 0$$



$$f_y(y) = \frac{f_X(x_1)}{|dg(x_1)/dx|} + \frac{f_X(x_2)}{|dg(x_2)/dx|} + \cdots$$

$$f_y(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}}$$

Markov inequality

- Markov's inequality gives an upper bound for the probability that a non-negative function of a random variable is greater than or equal to some positive constant.
- Markov's inequality (and other similar inequalities) relate probabilities to expectations, and provide (frequently) loose but still useful bounds for the cumulative distribution function of a random variable.

Markov inequality

- **if X is any random variable and a > 0, then**

$$\Pr(|X| \geq a) \leq \frac{\mathrm{E}(|X|)}{a}.$$

Proof:

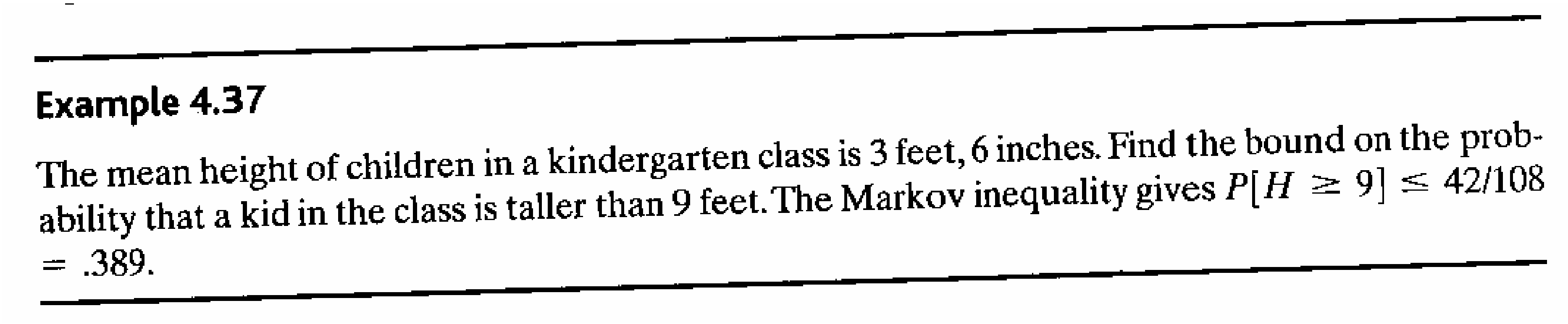
For a>0. let

$$I = \begin{cases} 1, & \text{if } X \geq a \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Since} \quad X \geq 0, \quad I \leq \frac{X}{a}$$

Taking expectation of the above $$E[I] \leq \frac{E[X]}{a}$$

$$\text{But} \quad E[I] = \Pr\{X \geq a\}, \qquad \therefore \Pr\{X \geq a\} \leq \frac{E[X]}{a}$$



Example 4.37

The mean height of children in a kindergarten class is 3 feet, 6 inches. Find the bound on the probability that a kid in the class is taller than 9 feet. The Markov inequality gives $P[H \geq 9] \leq 42/108 = .389$.

Chebyshev inequality

- Chebyshev inequality states that in any data sample or probability distribution, nearly all the values are close to the mean value, and provides a quantitative description of "nearly all" and "close to".
- In particular, $\Pr(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$
    - No more than 1/4 of the values are more than 2 standard deviations away from the mean;
    - No more than 1/9 are more than 3 standard deviations away;
    - No more than 1/25 are more than 5 standard deviations away;
    - and so on. In general:
    - No more than 1/k2 of the values are more than k standard deviations away from the mean.

Chebyshev’s Inequality

$$\Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

If X is a random variable with finite mean μ and variance σ^2, then for any value *k* > 0.

$$\Pr\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Proof: Since (*X* - μ)^2 is a non-negative random variable,

$$\Pr\{(X - \mu)^2 \geq k^2\} \leq \frac{E\left[(X - \mu)^2\right]}{k^2}$$

Markov inequality

$$\Pr\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Ex 8.1:

The numbers of robots produced in a factory during a day is a random variable with mean 100. Describe the probability that the factory's production will not less than 120 in a day. Also if the variance is known to be 5, then describe the probability that the factory's production will be between 70 and 130 in a day.

Ans

Let *X* be a random variable of the numbers of robots produced in a day. By the Markov's inequality:

$$P\{X > 120\} \le \frac{E[X]}{120} = \frac{100}{120} = \frac{5}{6}$$

Therefore, the probability that the factory's production will not less than 120 in a day is not greater than 5/6.

$$\Pr(|X| \ge a) \le \frac{\mathrm{E}(|X|)}{a}.$$

By the Chebyshev's inequality:

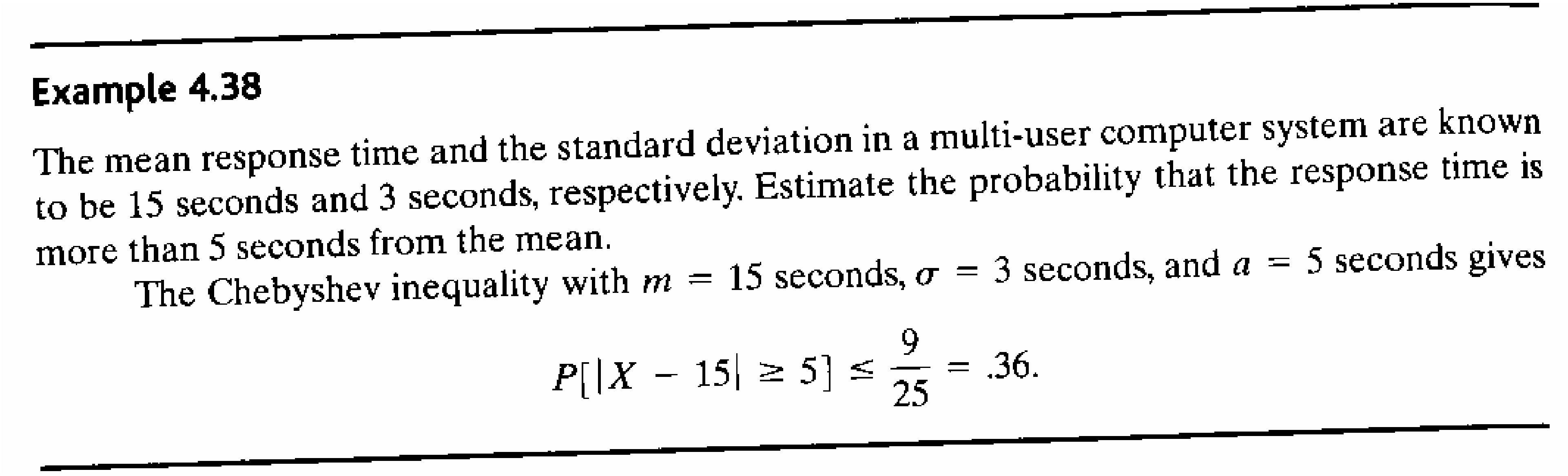
$$P\{|X - 100| \ge 30\} \le \frac{\sigma^2}{30^2} = \frac{1}{36}$$

$$P\{|X - 100| < 30\} \ge 1 - \frac{1}{36} = \frac{35}{36}$$

$$\Pr\{|X - \mu| \ge k\} \le \frac{\sigma^2}{k^2}$$

Therefore, the probability that the factory's production will be between 70 and 130 in a day is not smaller than 35/36.

$$\Pr\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$



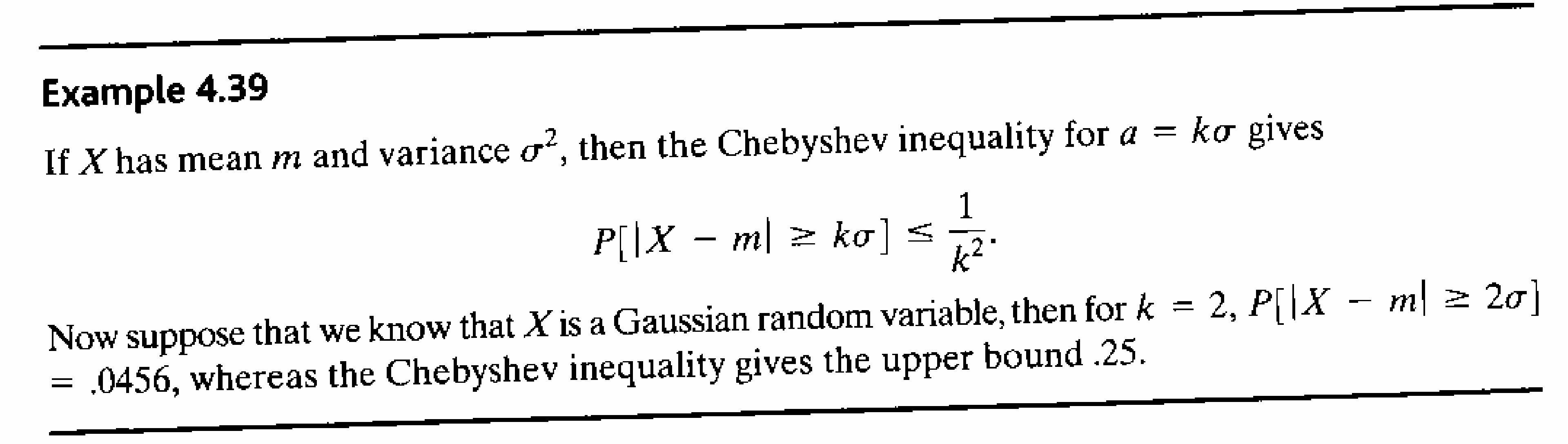
Example 4.38

The mean response time and the standard deviation in a multi-user computer system are known to be 15 seconds and 3 seconds, respectively. Estimate the probability that the response time is more than 5 seconds from the mean.

The Chebyshev inequality with $m = 15$ seconds, $\sigma = 3$ seconds, and $a = 5$ seconds gives

$$P[|X - 15| \geq 5] \leq \frac{9}{25} = .36.$$

$$\Pr\{|X-\mu|\geq k\}\leq\frac{\sigma^2}{k^2}$$



Example 4.39

If X has mean m and variance σ^2, then the Chebyshev inequality for $a = k\sigma$ gives

$$P[|X - m| \geq k\sigma] \leq \frac{1}{k^2}.$$

Now suppose that we know that X is a Gaussian random variable, then for $k = 2$, $P[|X - m| \geq 2\sigma] = .0456$, whereas the Chebyshev inequality gives the upper bound .25.

Transform Domain Methods

- These methods facilitates the analysis of random variables and their functions by looking at them in other ("frequency") domains

- We will consider two such methods:
  - Characteristic Function

Characteristic Function

- The Characteristic Function of a random variable X:

$$\Phi_x(\omega) = E\left[e^{j\omega x}\right]$$

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} e^{j\omega x} f_x(x)\,dx$$

- The characteristic function is the "Fourier transform" of the pdf (with sign reversal)

Characteristic Function

- The probability density function can be derived from the characteristic function:

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_x(\omega) e^{-j\omega x} \, d\omega$$

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} f_x(x) e^{j\omega x} \, dx$$

The pdf and the characteristic function form a unique "Fourier transform pair"

Example: Exponential RV

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} e^{j\omega x}\, f_x(x)\,dx$$

$$\Phi_x(\omega) = \int_{0}^{\infty} \lambda e^{-\lambda x}\, e^{j\omega x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{(j\omega - \lambda)x}\, dx = \frac{\lambda}{\lambda - j\omega}$$

Characteristic Function

- The characteristic function can also be defined for a discrete random variable

$$\Phi_x(\omega) = E\left[e^{j\omega x}\right]$$

$$\Phi_x(\omega) = \sum_k P_x(x_k)e^{j\omega x_k}$$

Characteristic Function

- In many cases, the discrete random variable X_k takes on integer values: $X_k=k$
- E.g. the number of “successes” in n Bernoulli trials

$$\Phi_x(\omega) = \sum_k P_x(x_k)e^{j\omega x_k}$$

$$\Phi_x(\omega) = \sum_k P_x(k)e^{j\omega k}$$

Characteristic Function

- Since

$$e^{j(\omega + 2\pi)k} = e^{j\omega k} e^{j2\pi k} = e^{j\omega k}$$

Therefore, $\Phi_X(\omega)$ for integer-valued (discrete) RV is a periodic function of ω

Characteristic Function

- For a discrete RV, the probability mass function can be derived from the characteristic function:

$$p_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k}\, d\omega \quad k=0,\ \pm 1,\ \pm 2,\ldots$$

$$\Phi_x(\omega) = \sum_k P_x(k) e^{j\omega k}$$

The pmf values $p_X(k)$ are the coefficients of the "Fourier series" of the (periodic)characteristic function $\Phi_X(\omega)$

Characteristic Function

- The Moment Theorem:

  The n^{th} moment of a random variable X can be evaluated using its characteristic function:

$$\boxed{E[x^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega)\Big|_{\omega = 0}}$$

Example: Exponential RV

- Let X be an exponential random variable with a pdf:

$$f_X(x) = \lambda e^{-\lambda x}$$

find the mean and variance of X using the moment theorem

$$\boxed{E[x^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega)\Big|_{\omega=0}}$$

Example: Exponential RV

- **Remember:**

$$\Phi_x(\omega) = \frac{\lambda}{\lambda - j\omega}$$

- **Therefore,**

$$\frac{d}{d\omega}\Phi_x(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}$$

Example: Exponential RV

$$\frac{d}{d\omega}\Phi_x(\omega) = \frac{\lambda j}{(\lambda - j\omega)^2}$$

$$\boxed{E[X] = \frac{1}{j}\frac{d}{d\omega}\Phi_x(\omega)\Big|_{\omega=0}}$$

$$E[X] = \frac{1}{j}\frac{j}{\lambda} \quad \Rightarrow \quad \boxed{E[X] = \frac{1}{\lambda}}$$

- Find the variance (as an exercise)