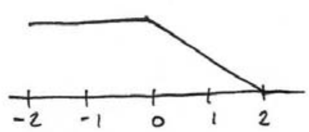
4,4 Important Continuous Random Variables

4.59

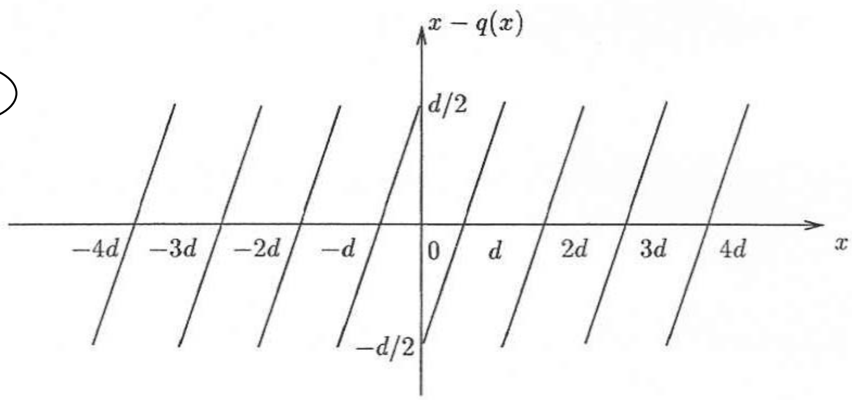
$$P[|X|>x] = P[\{X>x\} \cup \{X<-x\}]$$
$$= P[X>x] + P[X<-x]$$
$$= 1 - F_X(x) + F_X(-x)$$

$$F_X(x) = \begin{cases} 0 & x \le -2 \\ \frac{x+2}{4} & -2 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

$$P[|X|>x] = \begin{cases} 1 & x \le 0 \\ 1 - \left(\frac{x+2}{4}\right) + \left(\frac{-x+2}{4}\right) = 1 - \frac{x}{2} & 0 \le x \le 2 \\ 0 & x \ge 2 \end{cases}$$



4.60



for $-\frac{d}{2} < y < \frac{d}{2}$ the equation $y = x - q(x)$ has 8 roots, thus from Eqn. 3.55:

$$f_Y(y) = \sum_{k=1}^{8} \frac{f_X(x_k)}{\frac{dy}{dx}|_{x=x_k}}$$

Since $x - q(x)$ consists of piecewise linear unit-slope segments, we have that $\left.\frac{dy}{dx}\right|_{x=x_k} = 1$ all x_k.

Thus

$$f_Y(y) = \sum_{k=1}^{9} f_X(x_k) = \sum_{k=1}^{8} \frac{1}{8d} = \frac{1}{d}$$

for $-\frac{d}{2} < y < \frac{d}{2}$ ✓

5.29

a) For $0 \le y_0 \le x_0$ we integrate along the strip indicated below.

b) The marginal cdf's are obtained by taking the appropriate limits of the joint cdf:

$$
\begin{aligned}
F_X(x_0) &= \lim_{y_0 \to \infty} F_{XY}(x_0, y_0) = F_{XY}(x_0, x_0) = 1 - 2e^{-x_0} + e^{-2x_0} \\
F_Y(y_0) &= \lim_{x_0 \to \infty} F_{XY}(x_0, y_0) = 1 - e^{-2y_0}
\end{aligned}
$$

5.30

$$
\begin{aligned}
f_X(x) &= \int_0^\infty x e^{-x} e^{-xy} dy = x e^{-x} \left(\frac{-1}{x} e^{-xy} \right)_0^\infty = e^{-x} \\
f_Y(y) &= \int_0^\infty x e^{-x(1+y)} dx = \left. \frac{e^{-x(1+y)}((1+y)x - 1)}{(1+y)^2} \right|_0^\infty \\
&= \frac{1}{(1+y)^2}
\end{aligned}
$$

4.105

$$
\begin{aligned}
E[X] &= \frac{1}{j}\frac{d}{d\omega}e^{jm\omega-\sigma^2\omega^2/2}\Big|_{\omega=0} \\
&= \frac{1}{j}(jm-\sigma^2\omega)e^{jm\omega-\sigma^2\omega^2/2}\Big|_{\omega=0} \\
&= m \\
E[X^2] &= \frac{1}{j^2}\frac{d^2}{d\omega^2}e^{jm\omega-\sigma^2\omega^2/2}\Big|_{\omega=0} \\
&= \frac{1}{j^2}\left[-\sigma^2 e^{jm\omega-\sigma^2\omega^2/2}+(jm-\sigma^2\omega)^2 e^{jm\omega-\sigma^2\omega^2/2}\right]_{\omega=0} \\
&= \frac{1}{j^2}[-\sigma^2+j^2m^2] = \sigma^2+m^2 \\
VAR[X] &= E[X^2]-\mathcal{E}[X]^2 = \sigma^2
\end{aligned}
$$

4.106

$$
\begin{aligned}
\Phi_Y(\omega) &= E\left[e^{j\omega Y}\right] = E\left[e^{j\omega(aX+b)}\right] \\
&= E\left[e^{ja\omega X}\right]e^{j\omega b} \\
&= e^{j\omega b}\,\Phi_X(a\omega) \\
&= e^{j\omega b}\, e^{j\omega m-\frac{\sigma^2\omega^2}{2}}\Big|_{\omega=a\omega} \\
&= \underbrace{e^{j\omega b}\, e^{j\omega am-\frac{a^2\sigma^2\omega^2}{2}}}_{\text{characteristic fcn for Gaussian RV}}
\end{aligned}
$$

characteristic fcn for Gaussian RV
with mean $a\omega+b$
and variance $a^2\sigma^2$

5.9 Pairs of Jointly Gaussian Random Variables

5.110

$$f_{XY}(x,y) = \frac{e^{-(2x^2+y^2/2)}}{2\pi c}$$

The sol'n involves matching the coefficients of the polynomial in the exponent of the Gaussian pdf.

coeff. of $x^2 \Rightarrow \frac{1}{2(1-\rho^2)\sigma_1^2} = 2.$

coeff. of $y^2 \Rightarrow \frac{1}{2(1-\rho^2)\sigma_2^2} = \frac{1}{2}$

coeff of $xy \Rightarrow \frac{-2\rho}{2(1-\rho^2)\sigma_1\sigma_2} = 0 \quad \therefore \rho = 0$

$$\sigma_1^2 = \frac{1}{2(1-0^2)\cdot 2} = \frac{1}{4}$$

$$\sigma_2^2 = \frac{1}{2(1-0^2)\cdot\frac{1}{2}} = 1$$

$\therefore \text{COV}(X,Y) = \rho_{XY}\sigma_1\sigma_2 = 0$

$\text{VAR}[X] = \frac{1}{4}$

$\text{VAR}[Y] = 1$

5.6 Joint Moments and Expected Value of a Function of Two Random Variables

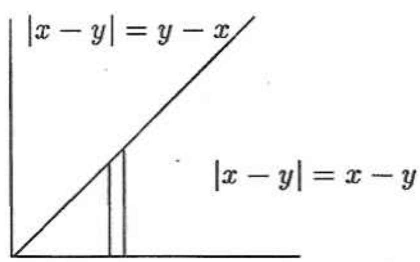
5.56

~~4.59~~ a) $\mathcal{E}[(X+Y)^2] = \mathcal{E}[X^2 + 2XY + Y^2] = \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2]$

b)
$$\begin{aligned} VAR[X+Y] &= \mathcal{E}[(X+Y)^2] - \mathcal{E}[X+Y]^2 \\ &= \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2] - \mathcal{E}[X]^2 \\ &\quad -2\mathcal{E}[X]\mathcal{E}[Y] - \mathcal{E}[Y]^2 \\ &= VAR[X] + VAR[Y] + 2[\mathcal{E}[XY] - \mathcal{E}[X]\mathcal{E}[Y]] \end{aligned}$$

c) $VAR[X+Y] = VAR[X] + VAR[Y]$ if $\mathcal{E}[XY] = \mathcal{E}[X]\mathcal{E}[Y]$ that is, if X and Y are <u>uncorrelated</u>.

5.57



$$\begin{aligned} \mathcal{E}[|X-Y|] &= \int_0^\infty \int_0^\infty 2\,|x-y| e^{-(2x+y)} dx\,dy \\ &= 2\int_0^\infty \int_0^\infty 2(x-y) e^{-2x} e^{-y} dy\,dx \\ &= 2\int_0^\infty 2\,e^{-2x} \underbrace{[x(1-e^{-x}) - \underbrace{\int_0^x y e^{-y} dy}_{1-(1+x)e^{-x}}}_{x+e^{-x}-1} \\ &= 2\int_0^\infty 2(xe^{-2x} + e^{-3x} - e^{-2x})dx \\ &= \cancel{2[1+\tfrac{1}{2}-1]}\quad 2\left[\frac{1}{2} + 2\cdot\frac{1}{3} - 1\right] = 1 + \frac{4}{3} - 2 = \frac{1}{3} \end{aligned}$$

5.58

5.58) $E[X^2 e^Y] = E[X^2]E[e^Y] = 1\cdot E[e^Y] = \frac{1}{3}\cdot(e^3-1)$

~~1 $\mathcal{E}[X^2Y] = \mathcal{E}[X^2]\mathcal{E}[Y] = 1(1) = 1$~~

$\frac{1}{3}\int_0^3 e^y dy = \frac{1}{3} e^y\big|_0^3 = \frac{1}{3}(e^3-1)$