EM Implosion Memos

Memo 28

June 2009

Derivation of the dielectric constant as a function of angle for designing a conical non-uniform launching lens

Prashanth Kumar, Carl E. Baum, Serhat Altunc, Christos G. Christodoulou and Edl Schamiloglu
University of New Mexico
Department of Electrical and Computer Engineering
Albuquerque, NM 87131

Abstract

The dielectric constant, ϵ_r, as a function of angle, θ', is derived for the lens boundary curve. Limitations of the derivation are stated. A methodology for discretizing the curve is presented.

1 Introduction

There are many ways to vary the dielectric constants in a non-uniform launching lens. The planar construction, perhaps the simplest, has been shown in [1]. However, the planar construction has the disadvantage that the ray paths within the lens are too complicated and hence simulation results are difficult to analyze. A natural way around the problem is to design the lens such that the dielectric constants are distributed along the ray paths i.e. each ray travels along a path with a unique dielectric constant. In such a design the equal time condition can be satisfied for all rays from the lens. Such a design is equivalent to stating that $\epsilon_r = f(\theta')$ (in the r', θ' coordinate system, [2]). However, the bending of rays due to $d\epsilon_r/d\theta'$ is neglected.

2 Derivation of ϵ_r as a function of θ'

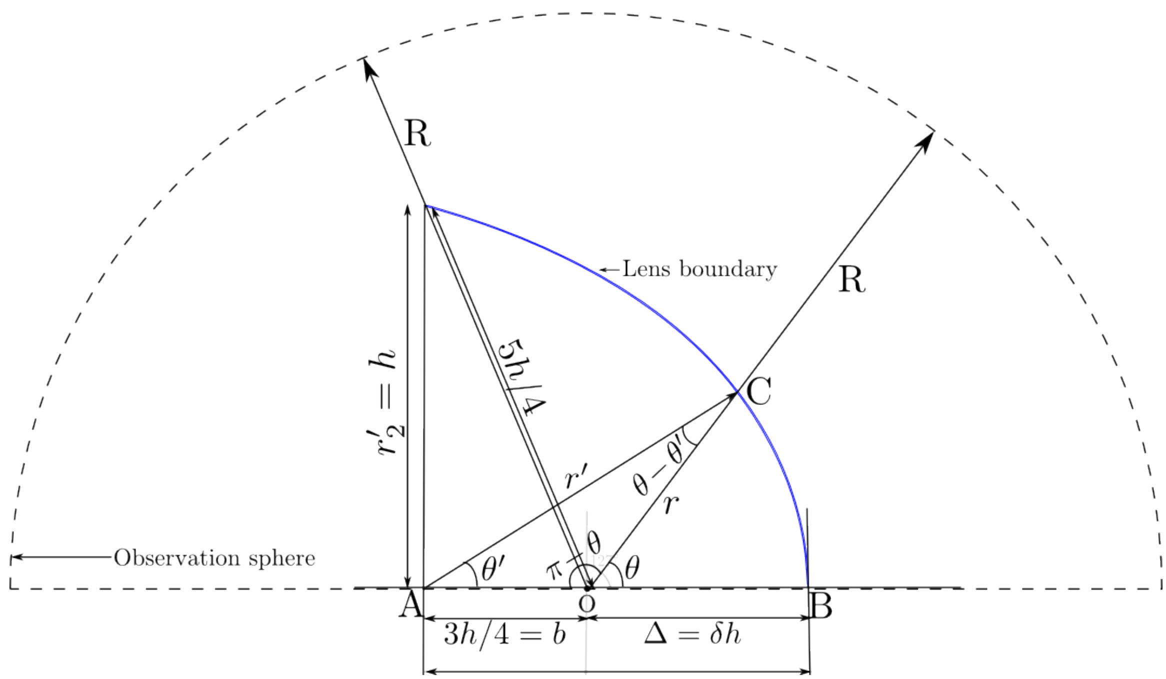

Figure 2.1: Diagram for deriving equation of ϵ_r as a function of θ' for an arbitrary lens boundary defined by the function $r'(\theta')$

In Fig. 2.1, O is the focal point and R is the radius of a sphere with O as center. The sphere is where the waves are being measured. Applying the sine rule to triange AOC

$$\frac{r'}{\sin(\pi - \theta)} = \frac{r}{\sin\theta'} = \frac{b}{\sin(\theta - \theta')} \tag{2.1}$$

We can obtain b from the first and third relation as

$$b = r'\left(\frac{\sin\theta\cos\theta' - \cos\theta\sin\theta'}{\sin\theta}\right) \tag{2.2}$$

From which θ is obtained as

$$\cos\theta' - \cot\theta\sin\theta' = \frac{b}{r'} \Rightarrow \theta = \left|\cot^{-1}\left[\left(\cos\theta' - \frac{b}{r'}\right)\right]\frac{1}{\sin\theta'}\right| \tag{2.3}$$

The absolute value is required in the above equation as one can obtain two solutions for θ. We want r' to always be positive. Negative angles will give negative r' which is not the convention used in our calculations.

Using the first and second relation from equation (2.1), and equation (2.3), we can obtain an expression for r

$$r = r'\frac{\sin\theta'}{\sin\theta} \text{ for } 0 < \theta' \leq \theta/2; \quad r = \delta h = \Delta \text{ for } \theta' = 0; \tag{2.4}$$

Since at $\theta' = 0$ we don't have a triangle and hence the sine rule is no longer valid.

Applying the equal time condition to a ray along r' and a ray along the height ($r'_2 = h$) of the lens

$$r'\sqrt{\epsilon_r(\theta')} + (R - r) = h\sqrt{\epsilon_r(\pi/2)} + \left(R - \frac{5h}{4}\right) \tag{2.5}$$

$\epsilon_r(\pi/2)$ is the dielectric constant at $\theta' = \pi/2$ i.e. the first layer (layer closest to switch/source). Hence,

$$\epsilon_r(\theta') = \left[\frac{h\sqrt{\epsilon_r(\pi/2)} + (r - 5h/4)}{r'}\right]^2 \tag{2.6}$$

where the function $r'(\theta') = a_0 + a_1\theta' + a_2\theta'^2 + a_3\theta'^3$ is defined in [2].

The incident angle θ_i as evaluated in [2] is

$$\theta_i = \cot^{-1}\left(\frac{\sqrt{\epsilon_r(\pi/2)} - \cos\psi}{\sin\psi}\right) = \tan^{-1}\left(\frac{\sin\psi}{\sqrt{\epsilon_r(\pi/2)} - \cos\psi}\right); \quad \psi = 37^\circ; \tag{2.7}$$

The boundary condition for the derivative of $r'(\theta')$ at $\theta' = \pi/2$ is $r'(\theta') = r_2\tan\theta_i = r_2\left(\frac{\sin\psi}{\sqrt{\epsilon_r(\pi/2)} - \cos\psi}\right)$. The equations needed to be solved to evaluate the constants ($a_0 - a_3$) in $r'(\theta')$ are

$$r'(0) = r'_1; \quad r'(\pi/2) = r'_2,$$
$$\frac{dr'(\theta')}{d\theta'}|_{\theta'=0} = 0; \quad \frac{dr'(\theta')}{d\theta'}|_{\theta'=\pi/2} = r_2\left(\frac{\sin\psi}{\sqrt{\epsilon_r(\pi/2)} - \cos\psi}\right),$$

For $r'_1 = r'_2 = h$ and $\theta' = 0$ equation (2.6) reduces to

$$\sqrt{\epsilon_r(0)} = \frac{h\sqrt{\epsilon_r(\pi/2)} + (\delta h - 5h/4)}{h} \tag{2.8}$$

For $\sqrt{\epsilon_r(0)} = 1.5 \Rightarrow \epsilon_r(0) = 2.25$ and $\delta = 1/4$

$$1.5 = \sqrt{\epsilon_r(\pi/2)} + (1/4 - 5/4) \Rightarrow \epsilon_r(\pi/2) = 2.5^2 = 6.25 \tag{2.9}$$

Note that the dielectric constants for this case are significantly lower than the planar case in [1], which is practically desirable.

There are two important factors/phenomena not considered in the derivation above:

1. Snell's law is not applied at the boundary interface between the lens and air for the (virtual) rays originating from the focal point. Note that a straight ray (radius R) is assumed to orginate inside the lens and follow the *same* path outside. This is obviously not electromagnetically true as one should take into account Snell's law. However, this would complicate the derivation as it would require prior knowledge of the dielectric constant at the boundary interface – the function that we are trying to calculate! One could obtain simultaneous equations or perhaps use a recursive technique.

2. It would follow from the application of the Eikonal equation in electromagnetics to the current problem that a continuous distribution in the dielectric constant would cause the rays to follow a curved path instead of a straight path inside the lens. However, this may not be too much cause of worry, since for practical reasons the lens is discretized, so that within each layer of constant ϵ_r the ray paths would be straight lines. However, a strict analytical treatment would have to take into account the Eikonal equation.

Nevertheless, the above derivation serves as a good first approximation and starting point.

3 Discretization of $\epsilon_r(\theta')$

It is very difficult to construct a physical lens with a continuously varying dielectric constant as derived above. Therefore, we must discretize the curve into regions of equal dielectric constants. This would not only make it easier to construct but also easier to simulate.

The discretization of the lens curve can be done using the tolerable time difference. Let the tolerable time difference between any two waves arriving on the measurement sphere be δ (≤ 10 ps). We want to split the lens boundary into ϵ_r increments such that the time difference of two rays between adjacent layers is not more than δ. Referring to Fig. 3.1, the condition for this is

$$(l_1\sqrt{\epsilon_{r_k}} + r_1) - (l_2\sqrt{\epsilon_{r_k}} + r_2) = \delta \times (3 \times 10^8 \text{ m/s}) \tag{3.1}$$

ϵ_{r_k} is the dielectric constant of the k$^{\text{th}}$ layer. The difference in times to travel two distances (l_1, l_2) along the lens boundary does not exceed δ. Note that ϵ_{r_k} is the same for both l_1 and l_2 (representing the same layer).

Figure 3.2 shows the application of the discretization condition, equation (3.1), for $\delta = 7.5$ ps. The dielectric constants and angles of various layers for Fig. 3.2 are tabulated in table 1. Note that equation (3.1) implies that at least 7 layers are needed for $\delta = 7.5$ ps. Lesser number of layers would be required for larger δ values. The lens is a body of revolution, so that in three dimensions each layer will give a cone. The entire geometry of the lens in three dimensions will appear as a series of concentric cones.

The derivation for ϵ_r as a function of θ' presented above, although semi-analytical, should yield better results than the planar cases described in [1]. This is because all the rays from the lens satisfy the equal time condition and undergo minimum number of reflections.

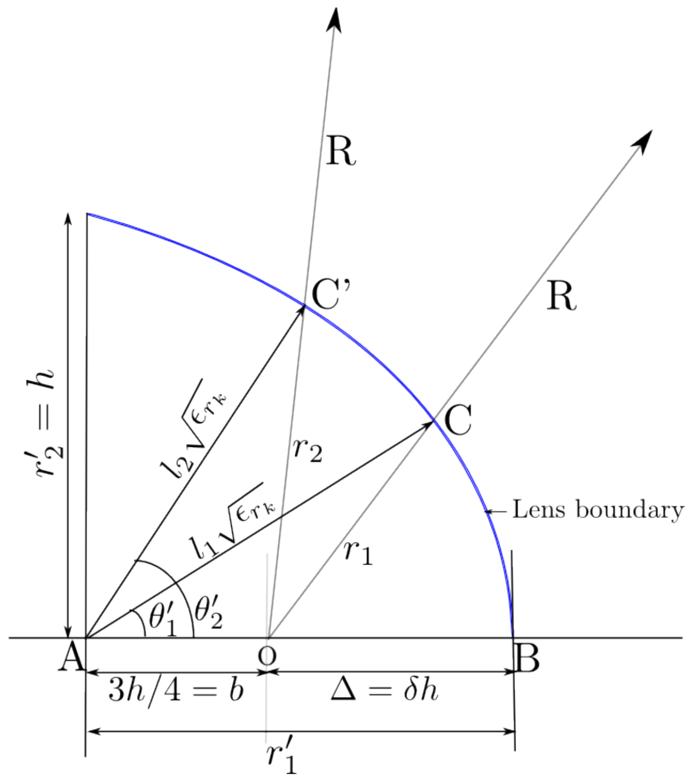

Figure 3.1: Diagram for discretizing $\epsilon_r(\theta')$ on the lens boundary.

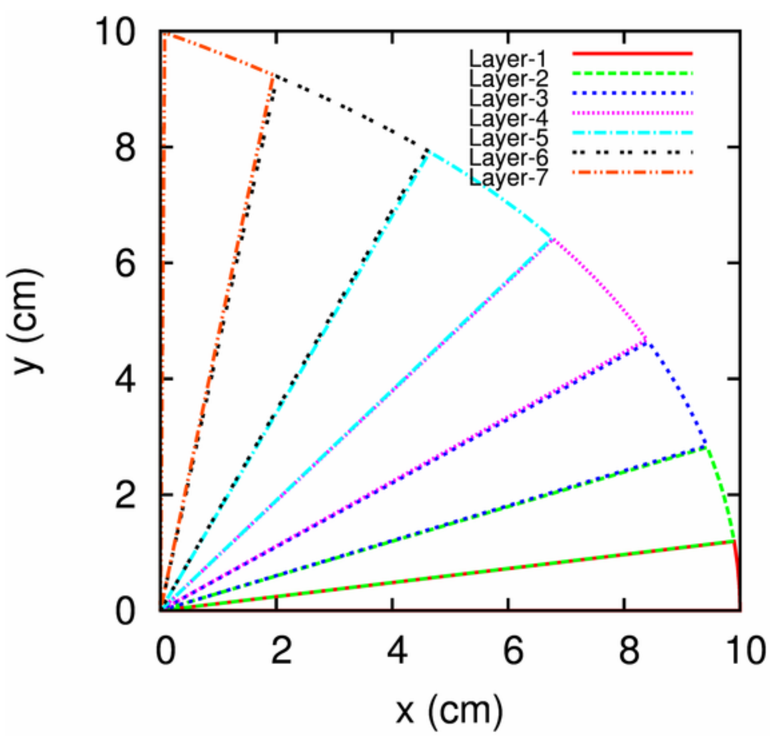

Figure 3.2: Plot of $\epsilon_r(\theta')$ curve discretized on the lens boundary.

Table 1: Dielectric constants and angles of various layers in Fig. 3.2

Dielectric constant	Angle in degrees
2.250	6.894
2.318	16.799
2.621	29.117
3.228	43.550
4.135	59.92
5.214	78.103
6.250	98.010

4 Conclusion

ϵ_r as a function of θ' has been derived for the lens boundary curve. Limitations of the derivation are stated. The curve is discretized so as to enable easier construction and simulation of the problem. In general, results for the conical case are considered more reliable as calculations presented here give an analytical basis for design and diagnosis of simulation and experimental results.

References

[1] Prashanth Kumar, Serhat Altunc, Carl E. Baum, Christos G. Christodoulou, Edl Schamiloglu, "Simulation results for 3-layer and 6-layer planar non-uniform launching lens." EM Implosion Memo 27, June 2009.

[2] Prashanth Kumar, Serhat Altunc, Carl E. Baum, Christos G. Christodoulou, Edl Schamiloglu, "Analytical considerations for curve defining boundary of a non-uniform launching lens." EM Implosion Memo 26, June 2009.