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RANDOM GRAPH MODEL FOR DETERMINING THE SURVIVABILITY OF SPATIALLY INHOMGENEOUS COMMUNICATION AND SENSOR NETWORKS

By

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ABSTRACT

We develop a model that shows how spatial inhomogeneity of node concentration influences the percolation threshold for a nearest neighbor Mobil Ad-hoc Network (MANET) system. The key elements of the method are based on the integration of a percolation threshold formula developed by Molloy and Reed [4] combined with random geometric graph theory. A sample calculation that illustrates the importance of key parameters is rendered.

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1. Introduction

Modern networks have large number of nodes which makes them robust against node failure under normal conditions. Of particular interest are MANETs that can be rapidly deployed for civilian and military applications. These nodes are randomly positioned and their statistical properties (e.g. average concentration, degree) are space dependent. However, High-Power Intentional Electromagnetic Interference (IEMI) pulses on sensor and information systems nodes can cause combinations of short-term upset, long-term upset and permanent damage to equipment. Any of these effects will degrade the performance of these systems. Mostly, these effects cause bit and message errors, but on some occasions physical damage also occurs. It is necessary to demonstrate their resilience against enemy attack by showing that connectedness is maintained at a sufficient information rate. Depending on the application, connectedness can range from minimum levels of survival awareness to being able to maintain a required communication rate over large segments of the network.

Over the past decade the property of connectedness—being able to sustain a network spanning cluster (NSC) has been demonstrated by several authors using combinations of random graph theory and percolation theory. A small set of references focused on the limited scope of this paper is rendered in [1-8]. References [3-8] are of general interest and have been useful in undirected graph theory models and in homogeneous networks. References [1, 2] provide a new theoretical principle that enable us to determine the percolation threshold—the fraction of surviving nodes that are required to sustain a network spanning cluster for hybrid undirected and directed graph models. Reference [1] and this paper show how to model heterogeneous communication and sensor networks. The combination of [1, 2] and the heterogeneous model of this paper allow one to address a general class of random geometric graphs that are characterized not by discrete location of nodes, but by continuous two dimensional spatial distributions. By combining these graphical representations in a common theoretical framework we provide a basis for expanding the theory to networks that are more closely connected to those frequently found in nature and in the military. The aforementioned methods allow one to make estimates of network survivability without incorporating unnecessary system-specific details.

This paper shows how spatial inhomogeneity influences the percolation threshold. For illustrative purposes we select the case where the nodes are coupled by nearest neighbor undirected links within a random geometric graph framework [6]. The starting point is the Reed and Molloy theoretical formalism based on nodes with discrete number of links [1, 4]. This is followed by using the mathematical properties of random geometric graphs. Several analytical models are then provided that show how the nodal two dimensional spatial density scale length affects the percolation threshold.

2. Brief Review of Network Spanning Cluster and Percolation Theory

Numerous authors have proposed that large mobile networks and MANETS could be modeled as random graphs and that survivability of these networks could be measured by their ability to support a network-spanning cluster (NSC) and the information rate. Depending on the situation, the required information rate that is useful for survivability could be measured in either bits/kilobits/megabits/etc. per second. Irrespective of the required information rate, being able to communicate over most of the physical size of the network with only a fraction of the nodes surviving is essential. This is the reason that so much importance has been attached to the ability of a network to form a NSC under adverse conditions. The mathematical parameter that defines the ability of a network to form a NSC is the percolation threshold p_c, which is computed in the following set of equations [1, 8].

The starting point is the normalized degree distribution function, $P(k)$, the number of edges, *k,* connected to a node. We have

$$1 = \sum_{k=2}^{k=k_{\max}} P(k) \tag{1}$$

where $k_{\max}$ is the maximum number of edges considered. Cohen et al [8] have shown that a network spanning cluster will be formed with a fraction, p_c, of the nodes given by

$$p_c = \frac{1}{K-1} \tag{2}$$

$$K = \frac{\langle k^2 \rangle}{\langle k \rangle} \tag{3}$$

To our knowledge we have demonstrated for the first time how to estimate the effects of two dimensional spatial variations of node density on the percolation threshold for a nearest neighbor MANET model. The key elements of the method are based on the integration of a percolation threshold formula developed by Reed and Molloy combined with random geometric graph theory. There remains the job of applying the model to actual case studies, especially those involving natural disasters.

$$\langle k \rangle = \sum_{k=2}^{k=k_{\max}} kP(k) \tag{4}$$

$$\langle k^2 \rangle = \sum_{k=2}^{k=k_{\max}} k^2 P(k) \tag{5}$$

Figure 1 depicts a hypothetical ad-hoc network for nearest neighbor communications and **Figure 2** shows a network spanning cluster of that network after attack. It is necessary to define the mathematical structure of the network of **Figure 1** under these conditions and the mathematical tools that are necessary to determine their survivability. Starting from this modest basis that each link (edge) had a fixed probability of survival, *p*, and there are *k* links connected to it one can compute the probability that the network will form a “network spanning cluster”. This cluster of nodes allows end-to-end communication at a minimum rate and thereby establishes a baseline level of survivability.

In order to calculate the percolation threshold using the formalism of equations (1) to (5) it is necessary to know the number of edges. This is often not readily available for mobile networks, and neither are the locations of all the nodes. More likely, what is known are local spatial averages of nodal surface concentrations (e.g., number of nodes per m^2). Consider for example the hasty construction of a mobile network following a natural disaster. We pose the question—can we achieve a network spanning cluster over a significant area? The mathematical model developed in the next section can answer this type of question.

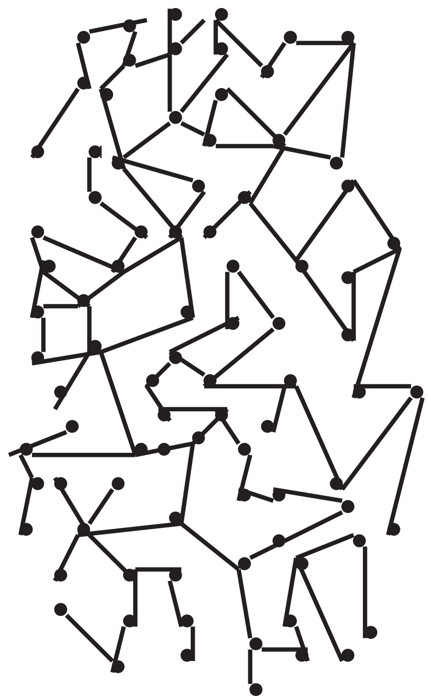

Figure 1 Hypothetical MANET before attack

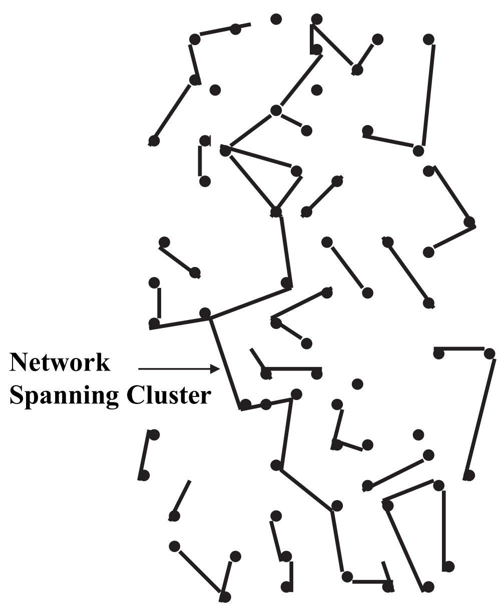

Figure 2 Network spanning cluster after attack

3. Analytical Approach to Spatial Inhomogeneous Network

The purpose of this section is to that shows how spatial inhomogeneity influences the percolation threshold. For illustrative purpose we select the case where the links are non-directed within a random geometric graph framework [6]. The starting point is Reed and Molloy's Q-function, given by

$$Q = \sum_{k\geq 1} k(k-2)P(k) \geq 0 \tag{6}$$

$$\sum_{k\geq 1} P(k) = 1 \tag{7}$$

In the foregoing expressions $P(k)$ is the probability that a node has k edges. The summations in equations (6) and (7) are over all nodes, and spatial issues never enter the picture explicitly. In contrast to the derivation by Cohen et al [8], as rendered in equations (1) to (5) equation (6) is derived by Reed and Molloy in a more comprehensive mathematical formalism. They showed that if there exist a sequence of nonnegative numbers $\lambda_1, \lambda_2, ...\lambda_M$ with M being some finite maximum number, and the condition of equation (8) applies we get

$$\sum_{k=1}^{k=M} \lambda_k = 1 \tag{8}$$

$$\sum_{k=1}^{k=M} k(k-2)\lambda_k > 0 \tag{9}$$

The connection to a survivable network is made by setting

$$\lambda_k \equiv P(k) \tag{10}$$

$$\sum_{k=1}^{k=M} k(k-2)P(k) > 0 \tag{11}$$

We rewrite equation (11) in the form

$$\sum_{k=1}^{k=M} k(k-2)P(k) = -P(1) + \sum_{k=2}^{k=M} k(k-2)P(k) > 0 \tag{12}$$

However, any node with only one edge cannot be counted as belonging to the network spanning cluster since there is no way this node can add connectedness. Therefore it can be removed: we set

$$P(1) = 0 \tag{13}$$

At first we are then left with the result

$$\sum_{k=2}^{k=M} k(k-2)P(k) > 0 \tag{14}$$

Each term of equation (14) is positive and in fact the minimum of the entire function is precisely zero. This occurs for the $k = 2$ case. The physical model for this special case is a linear chain of nodes; this is often realized in practice. Thus, the generalization of equation (14) is

$$\sum_{k=2}^{k=M} k(k-2)P(k) \geq 0 \tag{15}$$

$$\sum_{k=2}^{k=M} kkP(k) = \left\langle k^2 \right\rangle \geq 2\sum_{k=2}^{k=M} kP(k) = 2\left\langle k \right\rangle \tag{16}$$

There remains to show that equation (15) is equivalent to the set of equation (1) to (5). The only stiff requirement we have is that the percolation threshold must satisfied the condition

$$p_c = \frac{1}{K-1} = \frac{\left\langle k \right\rangle}{\left\langle k^2 \right\rangle - \left\langle k \right\rangle} \leq 1 \tag{17}$$

As observed, equation (17) is the same as equation (16). We are now ready to convert the foregoing discrete equations into continuous versions, sidetracking some of the familiar finer mathematical details with this process. Instead of equations (1) and (2) we now write

$$Q = \int_{k=2}^{\infty} k(k-2)dP(k) \geq 0 \tag{18}$$

$$\int_{k=2}^{\infty} dP(k) = 1 \tag{19}$$

Consider a system containing N nodes spread out over an area A_T in a plane defined by either Cartesian coordinates, $\vec{r} = (x, y)$ or by cylindrical coordinates $\vec{r} = (r, \varphi)$. Let $n(\vec{r})$ be the surface concentration of nodes. The total number of nodes in the system is

$$N = \iint n(\vec{r})dA = \iint n(\vec{r})rdrd\varphi \quad or \quad \iint n(\vec{r})dxdy \tag{20}$$

The spatial average nodal density is $n_0 = N / A_T$. Now let A_a be the area subtended by the node's communication antenna. We assume that all the nodes have the same communication system. For illustrative purposes let us assume that the transmitting and receiving antennas are omnidirectional. Now consider a point in the plane located at position: $\vec{r} = (r, \varphi)$. The density of nodes at $\vec{r}$ is defined by $n(\vec{r})$, and the range at which communication between two nodes is possible is defined by r_a. For an omnidirectional antenna we have the approximate relationship

$$A_a \cong \pi r_a^2 \tag{21}$$

The approximate nature of equation (21) is attributed to the fact that the signal-to-noise ratio (SNR) is somewhat variable due to the inherent variable nature of radio wave propagation. We assume that the variability of propagation range r_a is minor compared to other factors in the problem such as the spatial heterogeneity of node density—this is the major consideration. The foregoing analysis applies with good approximation when the scale length, η, for changes in nodal density spatial is much greater than r_a. That is,

$$r_a << \frac{n}{|\nabla n|} \approx \eta \tag{22}$$

We expect equation (22) to apply in large scale disasters and selected military applications.

When equation (22) applies the maximum number of nodes capable of coupling to the node located at position $\vec{r} = (r, \varphi)$ is $k_{\max} = n(\vec{r})A_a$. The number of available edges, *k*, is then

$$k = n(\vec{r})pA_a \tag{23}$$

where *p* is the probability for the edge to exist. Equation (23) provides the analytical relationship between connectedness and spatial variation of density. For example, in the homogeneous case: $n(\vec{r}) = n_0$, we can write

$$dP(k) = f(k)dk \tag{24}$$

where $f(k)$ is the degree probability density function, as yet to be determined. Since $n(\vec{r}) = n_0$ the degree is constant,

$$k_0 = n_0 pA_a \tag{25}$$

The degree probability density function is a delta function

$$f(k) = \delta(k - n_0 pA_a) = \delta(k - k_0) \tag{26}$$

We get the simple result

$$Q = k_0(k_0 - 2) \tag{27}$$

which shows that it is necessary for $k_0 \geq 2$ to get a network spanning cluster.

When $n(\vec{r})$ is not uniform the differential probability *dP* must also account for the spatial weighting of the nodes. We now write

$$dP = f(k,n)dkd\Gamma \tag{28}$$

$$d\Gamma = \frac{n(\vec{r})dA}{N} = \text{Differential fraction of nodes contributing to integrals} \tag{29}$$

We also need to ensure the condition of equation (19).

$$\int_{k=2}^{\infty} dP(k) = \int \int_{k=2}^{\infty} f(k,n)dkd\Gamma = \frac{1}{N}\int \int_{k=2}^{\infty} f(k,n)dkn(\vec{r})dA = 1 \tag{30}$$

When $k = n(\vec{r})pA_a$ applies we can use the result

$$f(k,n) = \delta(k - n(\vec{r})pA_a) \tag{31}$$

Since there is now a unique relationship between k and $n(\vec{r})$ we can write

$$f(k,n) = \delta(k - n(\vec{r})pA_a) \tag{32}$$

$$Q = \int k(k-2)dP(k) = \int\int k(k-2)f(k,n)dkdn = \frac{1}{N}\int npA_a(npA_a - 2)ndA \geq 0 \tag{33}$$

Equation (33) is the basis of deciding whether an inhomogeneous spatial distribution of nodes can support a network spanning cluster. We require

$$\int npA_a(npA_a - 2)ndA \geq 0 \tag{34}$$

The condition can also be expressed as

$$pA_a \int n^3 dA \geq 2\int n^2 dA \tag{35}$$

For illustrative example we neglect the angular dependence of node density and use the expressions:

$$n = Br^n \exp - \alpha r \tag{36}$$

$$dA = 2\pi r dr \tag{37}$$

The calculations are

$$pA_a \int n^3 dA = 2\pi pA_a B^3 \int_0^\infty \exp(-3\alpha r) r^{3n+1} dr = \frac{2\pi pA_a B^3}{(3\alpha)^{2+3n}} \int_0^\infty \exp(-x) x^{3n+1} dx = \frac{2\pi pA_a B^3 (3n+1)!}{(3\alpha)^{2+3n}} \tag{38}$$

$$2\int n^2 dA = 4\pi B^2 \int_0^\infty \exp(-2\alpha r) r^{2n+1} dr = \frac{4\pi B^2}{(2\alpha)^2} \int_0^\infty \exp(-x) x^{2n+1} dx = \frac{4\pi B^2 (2n+1)!}{(2\alpha)^{2+2n}} \tag{39}$$

Equation (35), the percolation threshold condition becomes

$$\frac{pA_a B}{2\alpha^n}\left(\frac{(3n+1)!}{(2n+1)!}\frac{2^{2+2n}}{3^{2+3n}}\right) \geq 1 \tag{40}$$

4. Way Forward

To our knowledge we have demonstrated for the first time how to estimate the effects of two dimensional spatial variations of node density on the percolation threshold for a nearest neighbor MANET model. The key elements of the method are based on the integration of a percolation threshold formula developed by Reed and Molloy combined with random geometric graph theory. There remains the job of applying the model to actual case studies, especially those involving natural disasters.

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