Probability and Statistics Notes

Note 9

The Confidence in Combinations of Imperfectly Known Variances

by

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Abstract

Evidence is presented in support of the following conjecture: Let $x_i : N(0,\sigma_i^2)$, where $i \in \{1,\ldots,N\}$, and let $[-e_i,e_i]$ be an R-reliable interval for x_i, at confidence level $\tilde{C}$. Then the RSS of these intervals will be an R-reliable interval for $X = \sum_{i=1}^{N} x_i$ at confidence level $C \geq \tilde{C}$.

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Introduction

Studies have shown that the RSS treatment (essentially addition of variances) is sometimes appropriate to error distributions and to the manner in which variates contribute to net error. A point at issue, however, is the effect on confidence of the RSS combination; i.e., if several 90-90 intervals are RSS combined, and the result is taken as covering 90% of the population, then what is the confidence in that result? An important aspect of this question is whether the resulting confidence (whatever value it is) is larger or smaller than the initial confidence. If it is larger one may combine large numbers of intervals and still retain an assured confidence in the end result. If the net confidence is lower than that of the individual intervals, then it becomes imperative to compute its value to know what confidence to place in the net error interval. The latter situation can cause significant problems on most engineering programs.

The following paragraphs demonstrate that the answer to this question is a function of the distribution of the error interval. For the most commonly encountered distribution, the normal, the answer is that the confidence grows as estimates are combined. Distributions can be postulated which cause confidence to decline.

a. Samples Drawn from a Normal Population

For the case of a normal population, the estimate of an error interval is within a known constant multiple of an estimate of the standard deviation. For convenience, the discussion below deals with estimates of the standard deviation. The discussion also deals with the combination of two error intervals; the process is readily extended.

Let $e_{i1}, e_{i2}, \ldots, e_{iN_i}$ $i = 1,2$ be a random sample ordered from smallest to largest from the normal population E_i :

$$E_i : N(o, \sigma_i^2) \quad i = 1,2 \tag{1}$$

The range of the samples is a random variable,

$$\omega_i = e_{iN_i} - e_{i1} \quad i = 1,2 \tag{2}$$

Let the standardized range be defined by

$$W \triangleq \frac{\omega}{\sigma} \tag{3}$$

where

$\sigma \triangleq$ standard deviation of the sampled population.

For a sample of size N, W has the known* and tabulated** probability density function f(W,N),

$$f(W,N) = \int_0^W [\Phi(W + X) - \Phi(X)]^{N-2} \phi(W + X)\phi(X)dX$$

$$\phi(X) \triangleq \frac{1}{\sqrt{2\pi}} e^{\frac{X^2}{2}}$$

$$\Phi(W) \triangleq \int_0^W \phi(X)dX$$

*Reference 1
**Reference 2

The variate W has the known* and tabulated** cumulative probability function $F(W,N)$

$$F(W,N) = \int_0^W f(Y,N)dY$$

By definition

$$Pr(W \leq W_o | N) = F(W_o,N)$$

Where there is no question about the value of N we shall write more simply,

$$Pr(W \leq W_o) = F(W_o)$$

It is convenient to define $W_{\alpha N}$ by

$$F(W_{\alpha N}, N) = \alpha \qquad (4)$$

where $\alpha \triangleq$ selected probability value. An estimate S_α of the population standard deviation σ can be calculated from a sample with range ω by

$$S_\alpha = \frac{\omega}{W_{\alpha N}}$$

The confidence in S_α is found as follows. By Equations (3) and (4),

$$Pr(\frac{\omega}{\sigma} \leq W_{\alpha N}) = \alpha$$

$$Pr(\frac{\omega}{W_{\alpha N}} \leq \sigma) = \alpha$$

$$Pr(S_\alpha \leq \sigma) = \alpha$$

$$C(S_\alpha) \triangleq Pr(\sigma \leq S_\alpha) = 1 - \alpha$$

*Reference 1
**Reference 2

Recalling equations (1) and (2), we therefore define

$$S_{\alpha i} = \frac{\omega_i}{W_{\alpha Ni}} \qquad i = 1, 2$$

and it follows that

$$C(S_{\alpha i}) = Pr(\sigma_i \leq S_{\alpha i}) = 1 - \alpha$$

This equation provides the confidence in one specific value of $S_{\alpha i}$ (the one to be computed when a specific value of ω_i is obtained), but the quantity to be needed later is the general distribution function of $S_{\alpha i}$. This is found as follows:

Let K_i be any positive real number. Compute $W_{\beta Ni}$ by

$$W_{\beta Ni} = K_i W_{\alpha Ni}$$

where β is defined by

$$Pr\left(\frac{\omega_i}{\sigma_i} < W_{\beta Ni}\right) = \beta$$

It follows that

$$Pr\left(\frac{\omega_i}{W_{\alpha Ni}} < K_i\sigma_i\right) = \beta = F(W_{\beta Ni}, Ni)$$

$$Pr(S_{\alpha i} < K_i\sigma_i) = F(K_i W_{\alpha Ni}, Ni)$$

The cumulative distribution function $H_i(x)$ is thus

$$H_i(K_i\sigma_i) = F(K_i W_{\alpha Ni}, Ni) \tag{5}$$

and the probability density function $h_i(x)$ is,

$$h_i(x) = \frac{\partial}{\partial x}\left[F\left(\frac{XW_{\alpha Ni}}{\sigma_i}, Ni\right)\right]$$

i.e.,

$$h_i(x) = \frac{W_{\alpha Ni}}{\sigma_i} f\left(\frac{XW_{\alpha Ni}}{\sigma_i}, Ni\right) \qquad (6)$$

Let

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

be estimated by

$$S_\alpha^2 = S_{\alpha 1}^2 + S_{\alpha 2}^2$$

The confidence in S_α is

$$C(S_\alpha) = Pr(\sigma < S_\alpha) = P_r(\sigma^2 < S_{\alpha 1}^2 + S_{\alpha 2}^2)$$

Referring to Figure 1,

$$C(S_\alpha) = \iint_{RI} Pr(S_{\alpha 1}\ \varepsilon dS_{\alpha 1}\ \underline{\text{AND}}\ S_{\alpha 2}\ \varepsilon dS_{\alpha 2})$$

Changing to region II for convenience, and by the definition of the density function,

$$C(S_\alpha) = 1 - \iint_{RII} [h_1(S_{\alpha 1})dS_{\alpha 1}][h_2(S_{\alpha 2})dS_{\alpha 2}]$$

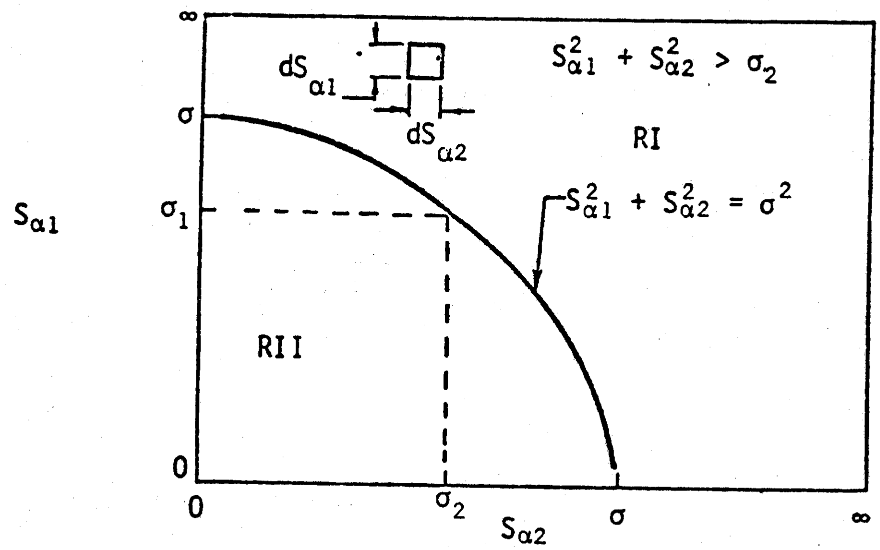

Figure 1. Definition of Regions for $S_{\alpha 1}$, $S_{\alpha 2}$

$$C(S_\alpha) = 1 - \int_0^{\sigma} \int_0^{\sqrt{\sigma^2 - S^2_{\alpha 2}}} h_1(S_{\alpha 1})\, h_2(S_{\alpha 2})\, dS_{\alpha 1}\, dS_{\alpha 2}$$

$$C(S_\alpha) = 1 - \int_0^{\sigma} \left[H_1\left(\sqrt{\sigma^2 - S^2_{\sigma 2}}\right) - H_1(0) \right] h_2(S_{\alpha 2})\, dS_{\alpha 2}$$

Noting that $H_1(0)$ is zero, we may re-write this equation as

$$C(S_\alpha) = 1 - \int_0^{\sigma} \left[H_1\left(\frac{\sqrt{\sigma^2 - S^2_{\alpha 2}}}{\sigma_1} \cdot \sigma_1 \right) \right] h_2(S_{\alpha 2})\, dS_{\alpha 2}$$

Utilizing equation (5) for the H_1 term and equation (6) for the h_2 term,

$$C(S_\alpha) = 1 - \int_0^{\sigma} F\left(\frac{\sqrt{\sigma^2 - S^2_{\alpha 2}}}{\sigma_1} W_{\alpha N_1}, N_1 \right) \frac{W_{\alpha N 2}}{\sigma_2} f\left(\frac{S_{\alpha 2} W_{\alpha N 2}}{\sigma_2}, N_2 \right) dS_{\alpha 2}$$

Define a new variable of integration

$$Z \triangleq \frac{W_{\alpha N2}}{\sigma_2} S_{\alpha 2}$$

Then

$$C(S_\alpha) = 1 - \int_0^{\frac{\sigma}{\sigma_2} W_{\alpha N2}} F\left(\sqrt{\sigma^2 - \frac{\sigma_2^2 Z^2}{W^2_{\alpha N2}}}\ \frac{W_{\alpha N1}}{\sigma_1},\ N_1\right) f\ (Z,\ N_2)\ dZ \qquad (7)$$

Define A by

$$\sigma_1 = A\sigma_2$$

and equation (7) becomes

$$C(S_\alpha) = 1 - \int_0^{W_{\alpha N2}\sqrt{A^2 + 1}} F\left(\sqrt{\frac{W^2_{\alpha N2}\ (A^2 + 1) - Z^2}{A}}\ \frac{W_{\alpha N1}}{W_{\alpha N2}},\ N_1\right) f\ (Z,\ N_2)\ dZ \qquad (8)$$

Equation (8) is the desired expression for $C(S_\alpha)$. Note that when N_1, N_2, and ~ are specified, $C(S_\alpha)$ is a function solely of A. By symmetry it is only necessary to consider $1 \leq A \leq \infty$.

Equation (8) has been evaluated numerically for the case $\alpha = 0.1$, $N_1 = N_2 = 16$ and for A = 1, 2, and 3. For example, at A = 1, $W_{(0.1)(16)} = 2.61$.

$$C(S_{0.1})_{\substack{A = 1 \\ N_1 = 16 = N_2}} = 1 - \int_0^{3.691} F\left(\sqrt{13.624 - Z^2}\right) f(Z)\, dZ \qquad (9)$$

Figure 2 shows the curves for the functions in equation (9). One minus the area under the bottom right hand curve is the confidence in S_α. Table 1 lists the results for A = 1, 2, and 3. The confidence in S_α is greater than the confidence in $S_{\alpha 1}$ and $S_{\alpha 2}$.

When A approaches ∞, (ie, $\sigma_1 >> \sigma_2$), the σ_2 estimate ($S_{\alpha 2}$), is essentially eliminated and $C(S_\alpha)$ becomes simply $C(S_{\alpha 1})$. This can be shown from equation (8) by noting (1) that A can be chosen large enough to make the F term within ε of α for all Z where f(Z) has non-trivial value and, (2) the integral of f(Z) over all non-trivial values is within ε of 1. Thus, for large enough A, $C(S_\alpha) \simeq i - \alpha = C(S_{\alpha 1})$, and hence the last line of Table 1.

b. Distributions Other Than Normal.

The question can be asked as to what the minimum confidence in S_α might be if the distribution of the sampled population is not normal. This can be estimated with the aid of Figure 3 as follows. For simplicity, let $\alpha = 0.1$ in this discussion.

If $S_{\alpha 1} > \sigma_1$ and $S_{\alpha 2} > \sigma_2$, then it is immediately necessary that $S_\alpha > \sigma$. This case occurs in region I (RI) of Figure 3. The probability that the computed values ($\overline{S}_{\alpha 1}$, $\overline{S}_{\alpha 2}$) lie in RI is simply $0.9 \times 0.9 = 0.81$. This is the smallest possible confidence in S_α; it is the lower limit which can be calculated by non-parametric methods. The correct confidence (as shown in the last section) is the probability of lying above the curve $S_{\alpha 1}^2 + S_{\alpha 2}^2 = \sigma^2$; this includes the 81 pct of RI and some portion of RIII. As a side point it can be observed that the largest possible confidence in S is one minus the probability of lying in RII; ie, $1 - (0.1)(0.1) = 0.99$.

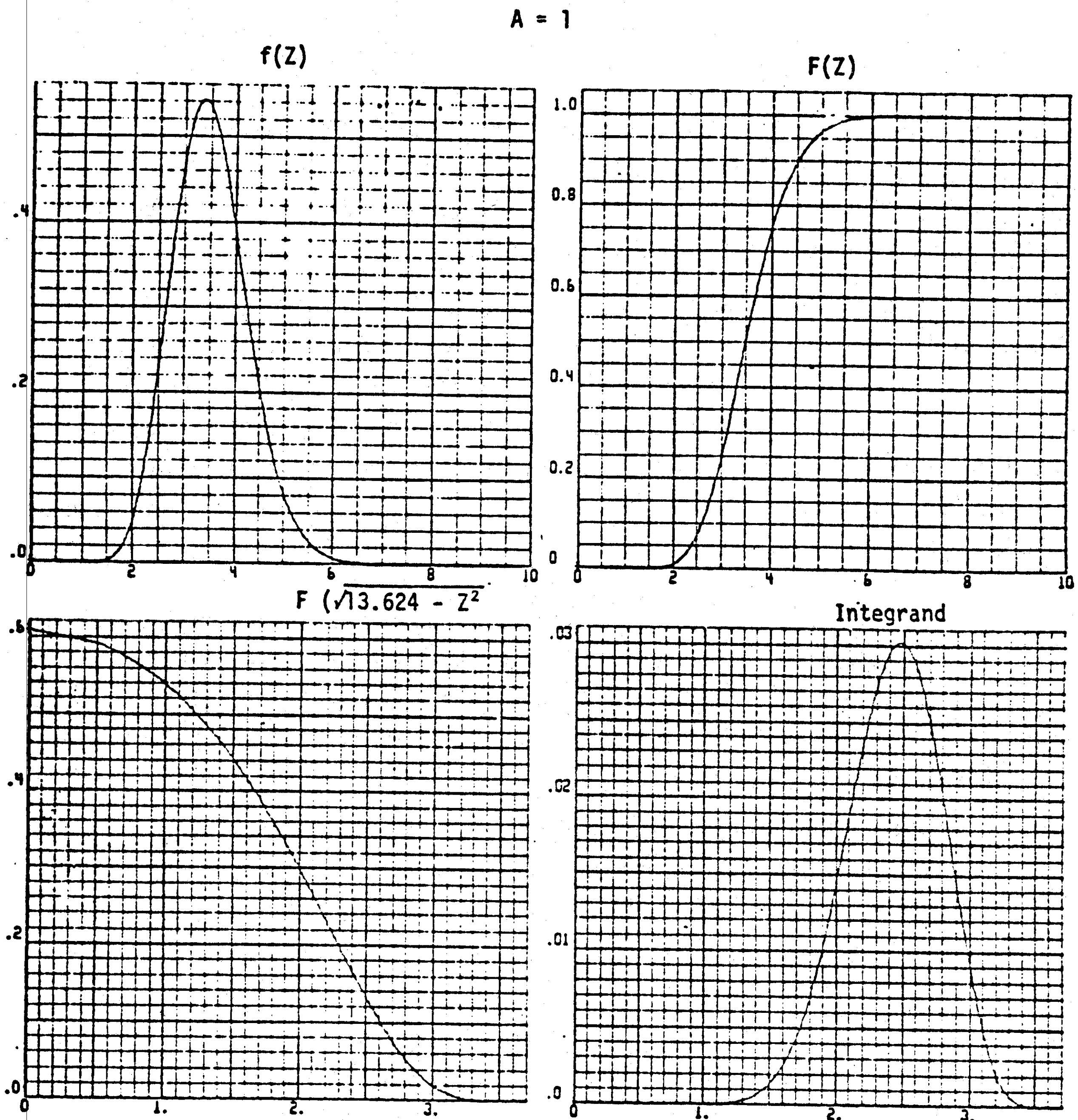

Figure 2. Curves for Functions in Equation (9).

TABLE 1

EXAMPLES OF CONFIDENCE IN S_α AS A FUNCTION OF A

$N_1 = N_2 = 16, \alpha = 0.1$			
A	$C(S_{\alpha1})$ (%)	$C(S_{\alpha2})$ (%)	$C(S_\alpha)$ (%)
1	90	90	97.24
2	90	90	05.20
3	90	90	93.05
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∞	90	90	90

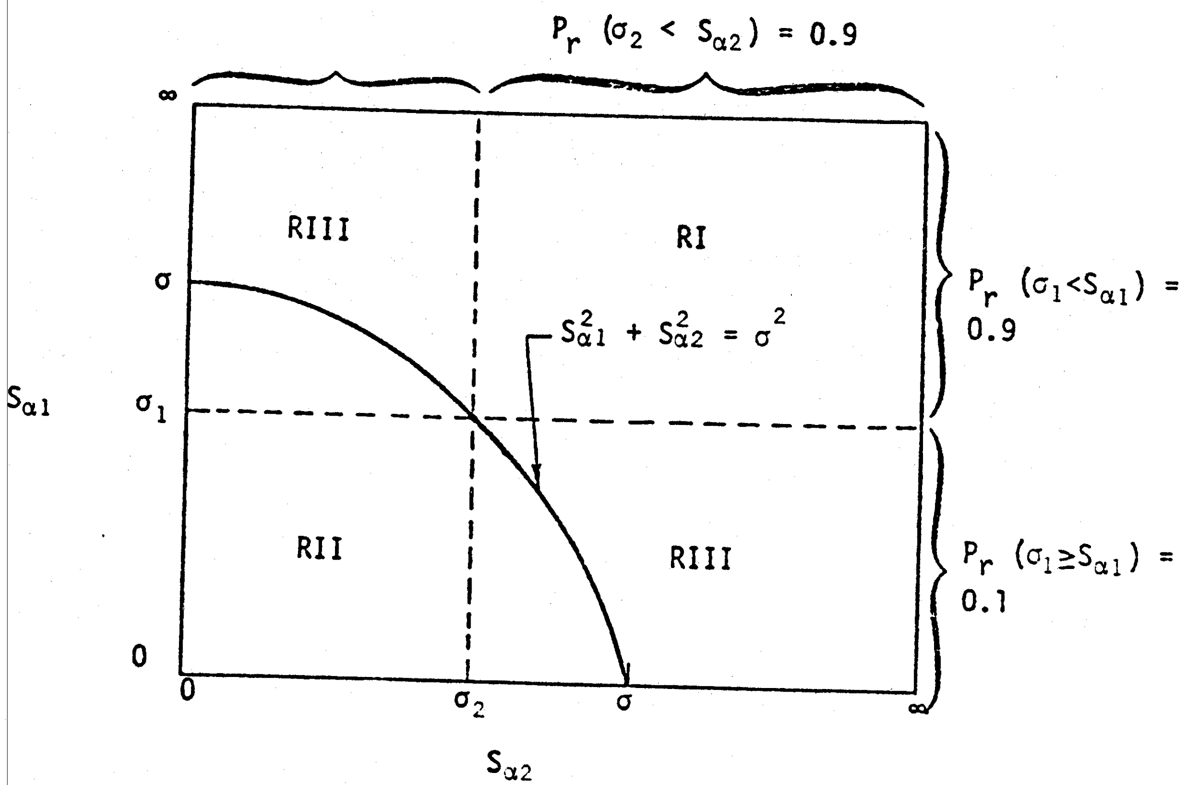

Figure 3. Probabilities of Different Regions of the $S_{\alpha1}$, $S_{\alpha2}$ Plane

It is possible to postulate a distribution for the $S_{\alpha 1}$ which leads to the minimum value of 81 pct. Let $S_{\alpha 1}$ be discrete random variable having only two possible values, A_1 and B_1; let $S_{\alpha 2}$ similarly have values A_2 and B_2. Let A_1 (and A_2) have probability 0.9, and B_1 (and B_2) have probability 0.1. Let A_1, B_1, A_2, and B_2 satisfy the following inequalities:

$$\sigma_1 < A_1 < \sigma$$

$$\sigma_2 < A_2 < \sigma$$

$$A_1^2 + B_2^2 < \sigma^2$$

$$A_2^2 + B_1^2 < \sigma^2$$

The four possible values for S_α and their associated probabilities are

$$({}_1S_\alpha)^2 = A_1^2 + A_2^2 \quad P_r({}_1S_\alpha) = 0.81$$

$$({}_2S_\alpha)^2 = A_1^2 + B_2^2 \quad P_r({}_2S_\alpha) = 0.09$$

$$({}_3S_\alpha)^2 = A_2^2 + B_1^2 \quad P_r({}_3S_\alpha) = 0.09$$

$$({}_4S_\alpha)^2 = B_1^2 + B_2^2 \quad P_r({}_4S_\alpha) = 0.01$$

From the inequalities corresponding to these equations, it is clear that only ${}_1S_\alpha$ is greater than σ; hence the confidence in S_α is 81 percent.

The example is simplest for discrete variables but can easily be cast into the form of continuous variables. The essential feature of this minimum confidence case is null at σ_i; a low or minimum confidence is obtained simply by minimizing the chances of lying in the upper and right portions of region III. When both variables are distributed as illustrated in Figure 4 the chances of lying in the upper part of RIII are small. A strongly bimodal distribution for the $S_{\alpha i}$ implies strongly bimodal distributions for the E_i error populations (see equation (1)).

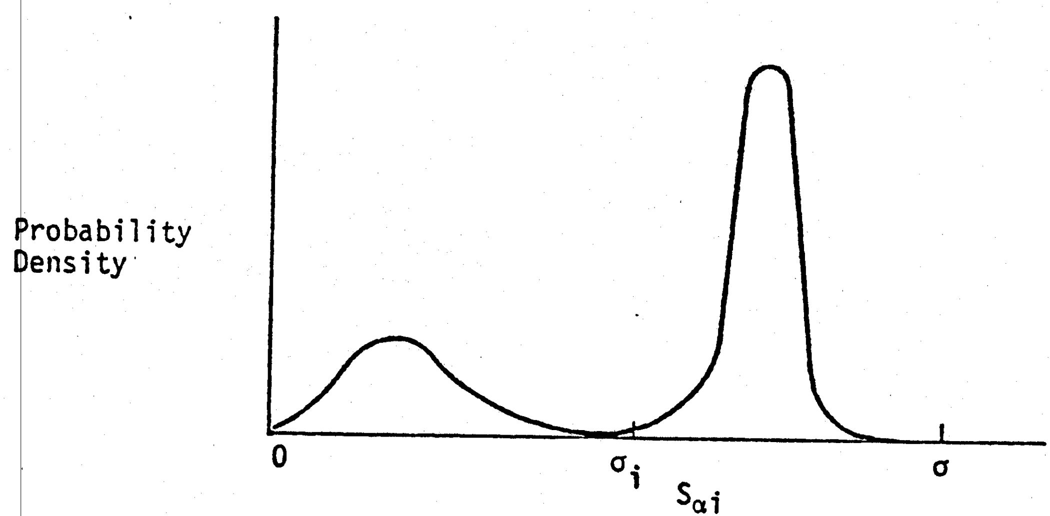

Figure 4. Distribution for $S_{\alpha i}$ for Low Confidence S

Conclusion.

The preceding two sections have shown that when two imperfectly known variances are added, the confidence in the result can be either larger (e.g., normally distributed populations) or smaller (e.g., bimodal populations). Distributions other than normal and strongly bimodal were not examined.

References.

1. *Statistical Theory and Engineering Applications*, by A. Hald. Wiley Publications, 1972.

2. "Order Statistics and Their Use in Testing and Estimation", by H. Leon Harter. Prepared for Aerospace Research Laboratories, Office of Aerospace Research, USAF, February 1969.