Probability and Statistics Notes

Note 6

Confidence in System Reliability

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Abstract

An approach to the problem of quantifying confidence in system reliability, as calculated from experimental data on system components, is presented. Some numerical results are calculated as examples.

Contents

Introduction

It is fairly common knowledge that if a system consists of a series of independent links, then the probability that the system will work is the probability that all the links will work, and that is the product of the individual link probabilities of working. Similarly, if a system consists of independent links arranged all in parallel with one another, then the probability of system failure is the probability that all the links will fail, and that is the product of the individual link probabilities of failure.

Unfortunately for these simple probabilistic network models, however, the individual link probabilities of success or failure can in real life never be known certainly. Instead, one usually has available some finite amount of success-failure data on links of the types in the system. This data leaves one more or less confident of any given suggested value of the probability of success or failure for a link. Consequently, when the proposed link probabilities are multiplied together there is some confidence (or lack of it) to be associated with the product.

In this note expressions are derived directly from basic definitions and axioms by means of which one can quantify such confidence.

(Equations (14), (17), and (23) of this note were first presented by the author in interoffice memoranda dated 5 and 6 October 1972.)

Some Notes on Confidence in System Reliability.

Part I. Preliminary definitions.

1. (System.) The word "system" will be used in this note to refer to a finite set of elements which has some purpose or job to perform. An example might be the set of components of a transmission system.

2. (N.) Let N be the number of elements in the system, i.e., the cardinality of the system. (Note that an element of a system is itself a subsystem, and that for the latter system $N = 1$.)

3. (To work.) In this note a system will be said to "perform successfully" or, more briefly, to "work", if it satisfies its purpose. Whether the purpose is satisfied is a judgement which must be made by the person who decides the purpose. An example might be that a transmission system must relay a unit of goods without damage, e.g., a unit of information without excessive distortion. (Note that when a system is an element in a larger system, then the purpose of the subsystem is to do whatever it must, given that sufficient other elements are working, to enable the larger system to perform successfully.) A system or element will be said to "fail" if it does not work.

4. (e_i.) Let e_i be the name of the i^{th} element in the system.

5. (E_i.) Assume that e_i has been selected at random for inclusion in the system from a population of candidate elements of that kind. Let E_i be the name of this candidate population. For example e_i might be a microwave relay link in an information transmission system selected randomly from a production line run E_i of many such links to be put into the i^{th} position in the system. (In this case one could have $E_i = E_j$ for $i \neq j$.) As another example, e_i might be a component of a transmission system in a particular state, where the component (and therefore the system) changes

states with time. Then the population E_i from which e_i is taken at a randomly selected instant is the set of instant-states (not necessarily distinct) in which that component might be at any instant. (In this latter example E_i has as many members as there are instants of time in which the component exists in one state or another.)

6. (N_i.) Let N_i be the number of elements in E_i, i.e., the cardinality of E_i.

7. (P_i.) Let P_i be the fraction of elements in E_i which work. (In the second example in paragraph 5, above, therefore, P_i is the fraction of time in which e_i works.) P_i is then the probability that a "randomly selected" member of E_i will be a working member.[1]

8. (L_i.) Let L_i be the number of elements of E_i which have been tested and ascertained either to work or not to work.

9. (M_i.) Let M_i be the number of elements of E_i which we have tested and ascertained to work. Thus,

$$M_i \le L_i \le N_i$$

for all $i \in \{1,2, \ldots ,N\}$.

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1. Note that P_i is a fixed, though unknown, number in the interval [0,1]. Its value is determined by the character of the population E_i, and not by either our data nor our state of information on the character of the population from any other source. All the usual concepts and theorems from probability theory apply to P_i. One of these concepts to which we will have occasion to refer later is that of probability independence (cf. Reference 1, p. 40, equation (2-40)).

10. ($C_i(p_k,p_\ell,L_i,M_i)$.) Let the confidence which one may reasonably have that

$$p_k \le P_i \le p_\ell$$

be represented by $C_i(p_k,p_\ell,L_i,M_i)$, where

$$0 \le p_k \le p_\ell \le 1 \quad .$$

By a "reasonable" confidence function we mean here one which satisfies three confidence axioms and a confidence requirement listed in paragraphs 10 and 11 of Reference 2.[2] It has been proved in earlier notes (cf. equations (4) in References 3 and 2) that

$$N_i < \infty \implies C_i(p,1,L_i,M_i) = \frac{\displaystyle\sum_{I=M_i}^{N_i(1-p)} \left[\binom{I}{L_i-M_i} \binom{N_i-I}{M_i} \right]}{\displaystyle\sum_{J=M_i}^{N_i-L_i+M_i} \left[\binom{J}{L_i-M_i} \binom{N_i-J}{M_i} \right]} \qquad (1)^3$$

and

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2. The reader may note here that we are drawing a distinction between probability and confidence, where the latter is, unlike the former, a function of our data, or general state of information about the population (cf. footnote 1, above). This distinction is what defines the dualist school of thought in probability theory: cf. Reference 4, p. 8, the paragraph beginning "Further historical note", and Reference 5, p. 622, the first through fourth complete paragraphs on the page. More detailed treatment of this issue is to be included in a forthcoming System Design and Assessment Note; additional references will be cited there.

3. The difference between the appearances of equation (1) in this note and equation (4) in Reference 3 is due to the difference in definitions of M.

$$N_i = \infty \Longrightarrow C_i(p,1,L_i,M_i) = \frac{\int_p^1 x^{M_i}(1-x)^{L_i-M_i}\,dx}{\int_0^1 x^{M_i}(1-x)^{L_i-M_i}\,dx} \qquad (2) .$$

From this point on in this note we will deal only with the case in which $N_i = \infty$ for all $i \in \{1,2, \ldots ,N\}$. We do this so as to have to develop only one sequence of formulas to illustrate the theory, because of the greater interest of the second example cited in paragraph 5, above, and because the corresponding formulas for the finite or mixed cases can easily be developed by the reader completely analogously merely by using equation (1) instead of equation (2) wherever appropriate. (The reader interested in the case of $N_i < \infty$ may also find Reference 6 helpful.)

11. ($f_i(p)$.) It will be helpful to consider the confidence density function $\frac{d}{dp}[C_i(0,p,L_i,M_i)]$, so we will give it a name: $f_i(p)$. That is,

$$f_i(p) \triangleq \frac{d}{dp}[C_i(0,p,L_i,M_i)] \qquad (3) .$$

Therefore, by the fundamental theorem of calculus (Barrow's Theorem),

$$C_i(p,1,L_i,M_i) = \int_p^1 f_i(x)\,dx \qquad (4) .$$

Equation (2) permits us to write immediately

$$f_i(p) = \frac{p^{M_i}(1-p)^{L_i-M_i}}{\int_0^1 x^{M_i}(1-x)^{L_i-M_i}\,dx} .$$

The denominator of this expression is just the Beta function[4], so the expression can be written more briefly as

$$f_i(p) = (L_i+1)\binom{L_i}{M_i} p^{M_i}(1-p)^{L_i-M_i} \qquad (5) .$$

12. (P.) Let P be the probability that the system will work at a given instant of time under given circumstances.

13. ($C(p_m,p_n)$.) We define $C(p_m,p_n)$ to be the confidence which one may reasonably have that

$$p_m \leq P \leq p_n ,$$

given the experimental evidence $\{L_1,M_1,L_2,M_2, \ldots ,L_N,M_N\}$. The purpose of this note is to develop a way of calculating $C(p_m,p_n)$. Confidence axiom III (cf. Reference 2, paragraph 11) tells us that

$$C((p_m,p_n) \text{ or } (p_n,1)) = C(p_m,p_n) + C(p_n,1)$$

for $p_m \leq p_n$. Therefore

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4. Cf. Reference 7: p. 342, equation 33; p. 344, equations (18.5.1) and (18.5.2); and p. 343, equation (18.3.2), third line. Or cf. Reference 8: p. 104, equation 367; and p. 103, equation 365, second line.

$$C(p_m,p_n) = C((p_m,p_n) \text{ or } (p_n,1)) - C(p_n,1) \equiv$$

$$= C((p_m,p_n) \cup (p_n,1)) - C(p_n,1) \equiv$$

$$= C(p_m,1) - C(p_n,1) \qquad (6) .$$

14. (C(R).) We will abbreviate C(R,1) by writing C(R) (R for system "reliability"; cf. Reference 3, paragraph 8 as far as equation (3)). That is,

$$C(R) \triangleq C(R,1) \qquad (7) .$$

There need be no confusion between the meanings of C as defined in this and in the preceding paragraph since the number of parameters listed makes clear which is intended. Combining equations (6) and (7), therefore, we have that

$$C(p_m,p_n) = C(p_m) - C(p_n) \qquad (8) .$$

The rest of the note will be devoted to presenting a way of calculating C(R) for some interesting kinds of systems, since with this and equation (8) one can readily calculate $C(p_m,p_n)$ for any $0 \leq p_m \leq p_n \leq 1$. By equation (4) we can already write

$$N = 1 \Longrightarrow C(R) = \int_R^1 f_1(p)\, dp \qquad (9)$$

(cf. equation (5)).

15. ($C_{i,j}(R_i,R_j)$.) Let $C_{i,j}(R_i,R_j)$ denote the confidence which we may reasonably have that

$$P_i \in [R_i,1] \quad \wedge \quad P_j \in [R_j,1] \qquad (10) .$$

By a "reasonable" confidence function we mean here, as usual, a function which satisfies the three confidence axioms and the confidence requirement (cf. Reference 2, paragraphs 10 and 11).

16. (Confidence independence.) We will not in this note seek to find a general expression for $C_{i,j}(R_i,R_j)$, since this would involve digressing into such concepts as conditional confidence. Rather, we will describe our knowledge of P_i as <u>independent</u> of our knowledge of P_j iff

$$C_{i,j}(R_i,R_j) = [C_i(R_i,1,L_i,M_i)] * [C_j(R_j,1,L_j,M_j)] \qquad (11) ,$$

and then restrict the discussion from this point forward to cases in which such independence is present. To paraphrase Papoulis (Reference 1, p. 40), "the reader might find this definition of independence arbitrary; he might even wonder if, in the case of real systems, there ever exists data satisfying both equation (11) and the confidence axioms and confidence requirement." Nonetheless we are in this note not going to undertake to develop exhaustive and rigorous tests for confidence independence. Rather, we will here leave it to the reader's engineering judgement to decide whether the data is such as to make his confidence in one system element's probability of working sufficiently independent of his confidence in another element's probability to make equation (11) an adequate model.

Part II. Series systems.

17. (Definition: series system.) A system will be said to be a "series system" iff it is necessary in order for the system to perform successfully that all of its elements work. As an example of a series system, consider the transmission system represented by the following figure:

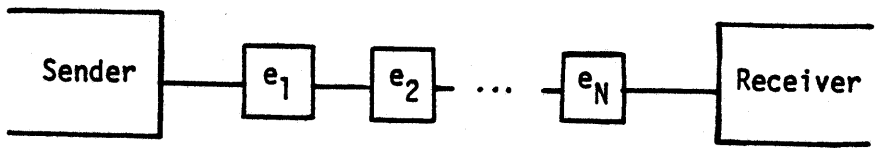

Figure 1. Series system.

An ordinary chain would be such a system, for transmitting a force: if any link fails to work, the chain won't work. A series of microwave relay links in an information transmission network would be another such system.

18. (Confidence in series system reliability.) For $N = 1$ equation (9) provides us with $C(R)$. For $N > 1$ we are going to assume that P_i is independent of P_j for all $i,j \in \{1,2, \ldots ,N\}$ and $i \neq j$ (cf. footnote 1, above)[5]. Therefore we may write

$$P = P_1 * P_2 * \ldots * P_N \quad .$$

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5. Invocation of this assumption means that the reader is obliged to satisfy himself that the elements in the real system of interest to him indeed have sufficiently independent probabilities of working (or of failing). Otherwise the model developed in this note may not be a good one for his system.

Consequently C(R) is our confidence that

$$R \le P_1 * P_2 * \ldots * P_N \le 1 \quad .$$

Consider first the particular case of N = 2. Then C(R) is our confidence that

$$R \le P_1 * P_2 \le 1 \qquad (12) \; .$$

There are several equivalent ways of writing this last expression. One which we can use is

$$P_1 \in [R,1] \quad \wedge \quad P_2 \in [\tfrac{R}{P_1},1] \qquad (13) \; .$$

(The reader should satisfy himself that expressions (12) and (13) are equivalent before proceeding.) Noting the similarity between expressions (13) and (10), then, we see that C(R), for a series system such that N = 2, is just $C_{1,2}(R,\frac{R}{P_1})$. Assuming confidence independence (with due regard for the analogue of footnote 5, above), and so invoking equation (11), and also equation (4), therefore, we have that

$$N = 2 \Longrightarrow C(R) = \int_{p_1=R}^{1} \int_{p_2=\frac{R}{p_1}}^{1} f_1(p_1)*f_2(p_2)\, dp_1 dp_2$$

(cf. equation (5)). Similarly C(R) for general N for series systems can be written as equation (14) (on the next page).

$$C(R) \;=\; \int_{p_1=R}^{1} \int_{p_2=\frac{R}{p_1}}^{1} \int_{p_3=\frac{R}{p_1 p_2}}^{1} \cdots \int_{p_N=\frac{R}{p_1 p_2 \cdots p_{N-1}}}^{1} f_1(p_1) f_2(p_2) f_3(p_3) \ldots f_N(p_N) \;\; dp_1 dp_2 dp_3 \ldots dp_N \qquad (14)\;.$$

(Cf. equation (5).)

Confidence in series system reliability.

19. (Example of calculation of confidence in series system reliability.) Let us apply equation (14) to a specific, simple, not uncommon example. Suppose a case has arisen such that testing has been uniformly successful, and the same number of samples were taken from all populations E_i.[6] That is,

$$L_1 = L_2 = \ldots = L_N = M_N = M_{N-1} = \ldots = M_2 = M_1 \quad .$$

Thus

$$f_1(p) = f_2(p) = \ldots = f_N(p) \quad .$$

Define

$$M = L \triangleq L_i = M_i$$

for $i \in \{1,2, \ldots ,N\}$. Then equation (5) becomes

$$f_i(p_i) = (L+1)p_i^M = (L+1)p_i^L = (M+1)p_i^M \qquad (15)$$

for all $i \in \{1,2, \ldots ,N\}$. Then equation (14) becomes

$$C(R) = (L+1)^N \int_{p_1=R}^{1} \cdots \int_{p_N=\frac{R}{p_1 \cdots p_{N-1}}}^{1} p_1^L \cdots p_N^L \; dp_1 \cdots dp_N \quad .$$

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6. One way in which this situation can arise is if all elements in the system were selected from some common population, for example microwave relay transceivers from a single stockpile of like transceivers. Then success-failure data on one link, to determine the fraction of time it spends in states in which it is vulnerable to upset, may in fact be applicable to all links, so that $L = \sum_{i=1}^{N} L_i$. Careful consideration must be given to the effects of the independence assumptions before proceeding to apply this model to these circumstances, however.

Let N = 2. That is, let the series system consist of only two series elements, as in the figure.

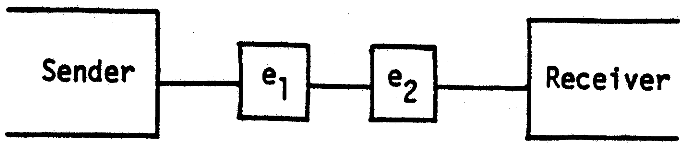

Figure 2. A series system.

Suppose L = 2 also. Then

$$C(R) = (2+1)^2 \int_{p_1=R}^{1} \int_{p_2=\frac{R}{p_1}}^{1} p_1^2 p_2^2 \; dp_1 dp_2 =$$

$$= 9 \int_{p_1=R}^{1} p_1^2 \left[\frac{p_2^3}{3} \right]_{\frac{R}{p_1}}^{1} dp_1 =$$

$$= 3 \int_{p_1=R}^{1} p_1^2 \left[1 - \left(\frac{R}{p_1} \right)^3 \right] dp_1 =$$

$$= 3 \left(\left[\frac{p_1^3}{3} \right]_R^1 - R^3 \left[\ln p_1 \right]_R^1 \right) =$$

$$= 1 - R^3 + 3R^3 \ln R = 1 - R^3 (1 - 3 \ln R) \qquad (16) .$$

Thus the confidence one may have that the circuit shown in Figure 2 is 50% reliable, assuming L = 2 and M = L, is

$$C(.5) = 1 - (.5)^3[1 - 3\,\ell n\,(.5)] \doteq 61.5069807\% \qquad (17)\ .$$

<u>Part III.</u> More definitions.

20. (Introduction to Part III.) The reader may note that this Part is quite similar to Part I, above. In the interest of brevity, therefore, and to avoid boring repetition, much of the analogous connective or explanatory material will be omitted this time. The reader who would like the help of such material is advised to refer to the corresponding definitions in Part I.

21. (Q_i.) Let Q_i be the fraction of elements in E_i which do not work. Q_i is then the probability that a "randomly selected" member of E_i will fail. Thus

$$Q_i = 1 - P_i$$

for all $i \in \{1,2, \ldots ,N\}$.

22. ($C_i'(q_k,q_\ell,L_i,M_i)$.) Let the confidence which one may reasonably have that

$$q_k \le Q_i \le q_\ell$$

be represented by $C_i'(q_k,q_\ell,L_i,M_i)$, where

$$0 \le q_k \le q_\ell \le 1 .$$

C_i' is related to C_i simply by reading "fail" for "work" everywhere, i.e., interchanging "M_i" and "L_i-M_i" everywhere. Therefore, by equation (4) in Reference 2,

$$C_i'(q_k,q_\ell,L_i,M_i) = \frac{\int_{q_k}^{q_\ell} x^{L_i-M_i}(1-x)^{M_i}\,dx}{\int_0^1 x^{L_i-M_i}(1-x)^{M_i}\,dx} \qquad (18) .$$

23. ($g_i(q)$.) Let

$$g_i(q) \triangleq \frac{d}{dq}\,[C_i'(0,q,L_i,M_i)] \qquad (19)\ .$$

Thus

$$C_i'(q,1,L_i,M_i) = \int_q^1 g_i(x)\,dx \qquad .$$

By equations (18) and (19),

$$g_i(q) = \frac{q^{L_i-M_i}(1-q)^{M_i}}{\int_0^1 x^{L_i-M_i}(1-x)^{M_i}\,dx} =$$

$$= (L_i+1)\binom{L_i}{L_i-M_i} q^{L_i-M_i}(1-q)^{M_i} \qquad (20)\ .$$

24. (Q.) Let Q be the probability that the system will fail at a given instant of time under given circumstances. Thus

$$Q = 1 - P \qquad .$$

Note that

$$P \geq R \iff -P \leq -R \iff$$

$$\iff 1 - P \leq 1 - R \iff$$

$$\iff Q \leq 1 - R \qquad (21)\ .$$

25. ($C'(q_m,q_n)$.) Let $C'(q_m,q_n)$ be the confidence which one may reasonably have that

$$q_m \leq Q \leq q_n \quad ,$$

given the experimental evidence $\{L_1,M_1,L_2,M_2, \ldots ,L_N,M_N\}$. As with equations (6),

$$C'(q_m,q_n) = C'(q_m,1) - C'(q_n,1) \quad .$$

By expression (21), and using confidence axiom II (cf. Reference 2, paragraph 11), therefore,

$$\begin{aligned} C(R) &= C(R,1) = \\ &= C'(0,1-R) = \\ &= C'(0,1) - C'(1-R,1) = \\ &= 1 - C'(1-R,1) \qquad (22) \;. \end{aligned}$$

Part IV. Parallel systems.

26. (Definition: parallel system.) A system will be said to be a "parallel system" iff it is sufficient for the system to work that <u>any</u> of its elements work. It is easy to see that an equivalent definition would be, a system is parallel iff it is necessary in order for the system to fail that all of its elements fail (cf. DeMorgan's laws). As an example of a parallel system, consider the transmission system represented by the following figure:

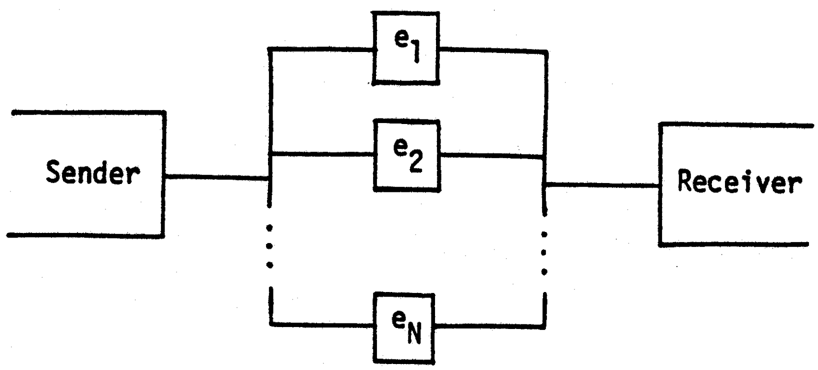

Figure 3. Parallel System.

27. (Confidence in parallel system reliability.) Comparison of the equivalent definition of a parallel system in the preceding paragraph with the definition of a series system given in paragraph 17 will show that the two definitions are formally identical, except that for parallel systems the word "fail" replaces the word "work" used for series systems. Consequently the assumptions, including those of independence, and reasoning which led to equation (14) yield a formally indistinguishable equation for C'(F,1) (F for "failure" bound) for parallel systems, mutatis mutandis. Therefore, by equation (22), C(R) for parallel systems can be written as equation (23) (on the next page).

$$C(R) = 1 - \left[\int_{q_1=1-R}^{1} \int_{q_2=\frac{1-R}{q_1}}^{1} \int_{q_3=\frac{1-R}{q_1 q_2}}^{1} \cdots \int_{q_N=\frac{1-R}{q_1 q_2 \cdots q_{N-1}}}^{1} g_1(q_1) g_2(q_2) g_3(q_3) \ldots g_N(q_N)\, dq_1 dq_2 dq_3 \ldots dq_N \right] \quad (23)\ .$$

(Cf. equation (20).)

Confidence in parallel system reliability.

28. (Examples of calculations of confidence in parallel system reliability.) Let us apply equation (23) to the same example we looked at in paragraph 19, above. That is, let N = 2, L = 2, and M = L, for the parallel system shown in the figure.

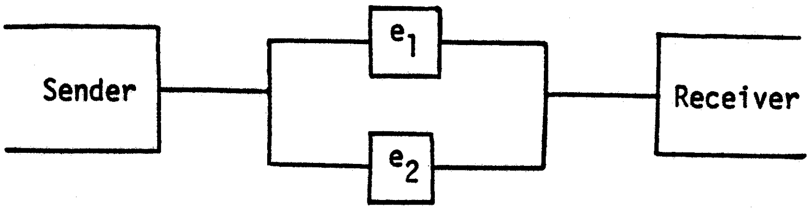

Figure 4. A parallel system.

Under these circumstances equation (20) becomes

$$g_i(q_i) = (L+1)(1-q_i)^M = (L+1)(1-q_i)^L = (M+1)(1-q_i)^M =$$
$$= 3(1-q_i)^2$$

for all $i \in \{1,2\}$. Then equation (23) yields

$$C(R) = 1 - \left[9 \int_{q_1=1-R}^{1} \int_{q_2=\frac{1-R}{q_1}}^{1} (1-q_1)^2(1-q_2)^2 \, dq_1 dq_2 \right] .$$

The integration can be done analytically. It is straightforward, but tedious, so we omit the details here. The result is

$$C(R) = 10(1-R)^3+9(1-R)^2-3(1-R)\{6+[(1-R)^2+6(1-R)+3]\ell n(1-R)\} .$$

Thus the confidence one may have that the circuit shown in Figure 4 is 50% reliable, assuming L = 2 and M = L, is

$$C(.5) = 10(.5)^3+9(.5)^2-3(.5)\{6+[(.5)^2+6(.5)+3]\ln(.5)\} \doteq$$

$$\doteq 99.8254817\%$$

This result should be compared with equation (17); it provides an interesting insight into the relationship between confidences in reliabilities of the circuits shown in Figures 2 and 4. Further insight into the relationship can be had by setting out to calculate confidence in reliability of the parallel system shown in Figure 4, using equation (23), but with L = 2 and M = 0. Equation (20) then becomes

$$g_i(q_i) = (L+1)q_i^L$$

for all $i \in \{1,2\}$. Note the similarity between this equation and equation (15). When one begins to evaluate equation (23) for R = .5 this similarity means one is simply repeating the work in equations (16). The result is that the confidence which one may have that the circuit shown in Figure 4 is 50% reliable, assuming L = 2 and M = 0, may be had using equation (17). Thus

$$C(.5) \doteq 1 - .615069807 = 38.4930193\%$$

Part V. Other systems.

29. Taken together, formulas (14) and (23) in this note are adequate for handling systems much more complicated than those represented by Figures 1 and 3. Consider, for example, the transmission system represented by the following figure:

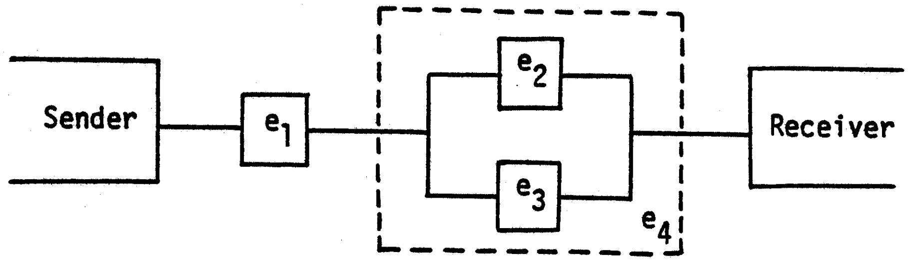

Figure 5. A series-parallel system.

Formula (23) can be applied to the part of the system inside the dashed box. The dashed box then has a known confidence function, so equation (8) applies. Treating the dashed box as a new single element e_4, one then knows $C_4(p_m, p_n)$. Differentiating this, numerically if necessary, by analogy with equation (3) yields $f_4(p)$. Formula (14) may then be used with $f_1(p_1)$ and $f_4(p_4)$ to yield C(R) for the series-parallel system. Many more complex finite systems composed of single-input-single-output independent elements can be treated in this way.

30. (Large N.) For small N there may be some hope of evaluating equations (14) and (23) analytically, as in the examples in paragraphs 19 and 28, above. However, for large N the integration in these formulas seems to become intractable very quickly. Approximate integration in many dimensions can still be achieved with the aid of a digital computer. Greater economy may be possible

by clever application of Monte Carlo techniques. (Cf. Reference 9, pp. 88 to 90. Cf. also Reference 10, especially pp. 191, 192, and 224.) Assumptions like that outlined in footnote 6, above, which yield $f_i = f_j$ and $g_i = g_j$ for many sets $\{i,j\} \subset \{1,2, \ldots ,N\}$, may lend even greater speed to a Monte Carlo algorithm.

Definitions of Symbols

Symbol	Meaning
$\triangleq$	is defined to mean
iff	if and only if
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by
$\wedge$	logical "AND"
$\equiv$	is identically the same as
$\doteq$	is approximately equal to
[a,b]	the closed interval from a to b
{a,b,...,z}	the set consisting of the elements a,b,...,z
ϵ	is a member of
U	union (of two sets)
$\subset$	is a subset of
*	"times" (i.e., ordinary multiplication; this operation is also indicated in some places by juxtaposition)
$\binom{a}{b}$	the binomial coefficient a over b (it is equal to $\frac{a!}{b!(a-b)!}$, where $a! \triangleq 1*2*3*\cdots*a$)

References

(These references are listed in the order in which they are first referred to in the text of the note.)

1. *Probability, Random Variables, and Stochastic Processes,* by Athanasios Papoulis. McGraw-Hill Book Company; copyright 1965.

2. *Confidence and Reliability in an Infinite Population,* AFWL EMP Probability and Statistics Note 2, 7 October 1971, by Chris Ashley.

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