Probability and Statistics Notes

Note 5

Confidence and Reliability in a Partitioned Finite Population

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Abstract

An approach to the problem of quantifying confidence in conclusions drawn from results of experiments is presented, given three conditions, viz., 1) dichotomy has been imposed upon the members of the population, 2) the cardinality of the population is finite, and 3) sampling is done randomly from distinguishable subsets which form a partition of the population. Some discussion is devoted to calculated numerical results.

PREFACE

Experiments are conducted in order to suggest and justify conclusions about the subject experimented upon. The greater the amount of supporting experimental data gathered, the higher the confidence one may reasonably have in conclusions drawn from that data. Thus, confidence is a monotone increasing function of the amount of experimental evidence supporting the conclusion. It is possible to quantify this confidence precisely.

In this paper it is shown by reasoning directly from this idea how to quantify confidence in the case where the population has been dichotomized and is finite and sampling has been done randomly within each of several distinguishable subpopulations. The only other assumption made is that prior to testing one is wholly ignorant of what fraction of the population is in either class of the dichotomy, meaning that he has equal confidence in all possible values of that fraction.

The analysis presented here assumes the reader is familiar with that in the paper, "Confidence and Reliability in a Finite Population" (cf. the Reference cited on the last page of this paper). If the reader is not familiar with the contents of that note, he is advised to read it before beginning to read this one.

A condition for applicability of the theory developed in the Reference is that the samples be selected randomly from the population as a whole. In some experiments this condition may not be satisfied. A sample might instead be random only within a particular subset of the population. If the sizes of the two dichotomy classes within the subset are independent of the sizes in the rest of the population, then the sample may give information about the composition of that subset only, and none about the rest of the population. The theory developed in the Reference may then be used to evaluate the information given about the subset, but a more general theory is needed in order to evaluate the implications of such sampling for the population as a whole. The present paper is intended to develop this more general theory.

(Between these extreme cases intermediary models may be more appropriate. For example, the sample might be random from a subset which was itself largely, but not entirely, randomly constructed relative to the dichotomy of interest. Then the sample would be random from the population as a whole to the same degree as was the subset, and so would give some lesser amount of information in addition about the rest of the population.)

More Notes on Confidence and Reliability in a Finite Population.

1. Let N be the number of elements in a certain set, and assume that $N < \infty$. An example of such a set is the set of Minuteman missiles. Assume also that the set has been dichotomized, i.e., that a criterion has been established which each element either satisfies or fails to satisfy. An example of such a criterion is that a Minuteman missile should be able to complete its mission despite an electromagnetic pulse environment. Assume it is possible to ascertain by test whether any particular element in the set is "good", i.e., satisfies the criterion, or not. Let L be the number of elements in the set which have been tested. In another paper (cf. the Reference cited on the last page of this paper) a detailed development of a theory is available covering the case in which the L tested elements were selected at random from the set as a whole. In this paper we wish to treat the case in which the elements were instead selected at random from within certain subsets, where the subsets partition the whole set. The number selected from within each subset need not be random; for example, all L may be selected from a single subset, perhaps the subset of those elements which are most cheaply accessible for testing. It is required only that, when once it was decided that a selection would be made from within a particular subset, then the selection was random from within that subset.

2. Let n be the number of subsets within each of which the selection for testing was random. Let N_i be the number of elements in the i^{th} subset, L_i the number chosen for test from the i^{th} subset, and M_i the number which failed the test from among the L_i. Thus

$$0 \le M_i \le L_i \le N_i \le N$$

for all $i \in \{1,2, \ldots ,n\}$. From these definitions it also follows immediately that

$$\sum_{i=1}^{n} N_i = N$$

and

$$\sum_{i=1}^{n} L_i = L \quad .$$

Let K be the (unknown) number of unsatisfactory elements in the set as a whole, and K_i the (unknown) number of unsatisfactory elements in the i^{th} subset. Assume the partitioning was done in such a way as to leave the K_i independent. Thus

$$K = \sum_{i=1}^{n} K_i \qquad (1) \quad .$$

Hence the number of elements which are satisfactory in the fleet as a whole is N-K, and the fraction of the set as a whole which is satisfactory is $\frac{N-K}{N}$. We desire to know, in light of the available experimental data, how confident we can reasonably be that this fraction is greater than or equal to a reliability value R. This confidence is of course a function of the value we choose for R. Therefore we seek an equation giving confidence C(R) that

$$\frac{N-K}{N} \geq R \qquad (2)$$

in terms of n, N_i, L_i, and M_i for $i \in \{1,2, \ldots ,n\}$.[1]

3. A complete description of the "real" state of the whole set, for our purposes, is given by the n-tuple $\{K_1,K_2, \ldots ,K_n\}$, since with this much information and the value of N we can use equation (1) to determine the truth or falsity of inequality (2). Pursuant to the approach developed in

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1. For n = 1 this equation should of course reduce to equation (4) in the Reference.

the Reference, we note that K_i might have any of the N_i+1 values between 0 and N_i, inclusive. Thus there are

$$A \triangleq (N_1+1)*(N_2+1)* \dots *(N_n+1) = \prod_{i=1}^{n} (N_i+1)$$

possible distinct n-tuples, and therefore A possible distinct states of such a partitioned set. For any one of these states we can calculate the number of ways in which we could have realized our experimental results $\{N_1,L_1,M_1,N_2, L_2,M_2, \dots ,N_n,L_n,M_n\}$. This we now do.

4. From equation (1) of the Reference we know that the number of ways of drawing L_i elements from the i^{th} subset, among which exactly M_i are unsatisfactory, given that K_i altogether are unsatisfactory in the N_i elements making up that i^{th} subset, is

$$I_i(K_i) = \binom{N_i-K_i}{L_i-M_i} * \binom{K_i}{M_i} .$$

Therefore the number of ways of getting the total experimental results $\{N_1,L_1,M_1,N_2,L_2,M_2, \dots ,N_n,L_n,M_n\}$, given the set state $\{K_1,K_2, \dots ,K_n\}$, is just

$$I(K_1,K_2, \dots ,K_n) = I_1(K_1) * I_2(K_2) * \dots * I_n(K_n) =$$

$$= \prod_{i=1}^{n} \left[\binom{N_i-K_i}{L_i-M_i} * \binom{K_i}{M_i} \right] \qquad (3)$$

(using the assumption that the K_i are independent).

5. Summing over all A possible states of the set, we have that the number of ways our experimental results could have been had from the set of all possible set states (i.e., the cardinality of the confidence sample space) is

$$D = \sum_{K_1=0}^{N_1} \sum_{K_2=0}^{N_2} \cdots \sum_{K_n=0}^{N_n} I(K_1,K_2, \ldots ,K_n) =$$

$$= \sum_{K_1=0}^{N_1} \sum_{K_2=0}^{N_2} \cdots \sum_{K_n=0}^{N_n} \prod_{i=1}^{n} \left[\binom{N_i-K_i}{L_i-M_i} * \binom{K_i}{M_i} \right] .$$

This will be the denominator in the confidence expression. However, as in paragraph 8 of the Reference, for a numerator we don't want to use the sum over all possible states, but rather the partial sum over all states $\{K_1,K_2, \ldots ,K_n\}$ such that equation (1) yields a K satisfying inequality (2). That is, the numerator is the partial sum of terms (given by equation (3)) over all values of K_i such that

$$\sum_{i=1}^{n} K_i \leq (1-R) * N = (1-R)\sum_{i=1}^{n} N_i .$$

6. Therefore C(R) is given by equation (4), on the next page. Computation time can be saved by noticing that the i^{th} factor in the product can be factored through the summation signs up to the i^{th} index of summation. We can also make use of the reasoning leading up to equation (2) in the Reference. Thus we obtain equation (5), on the next page. The reader may notice that equation (4) on p. 7 in the Reference is just the special case for which n = 1 of this more general expression.

7. Next the problem arises of evaluating equation (5). Paragraphs 9 and 10 of the Reference discuss this problem briefly, and offer one example of a computational result for the case in which n = 1. To evaluate the more general expression offered in equation (5) of this paper one might start by writing a function subroutine B such that

$$B(NI,MI,LI,MI) \triangleq \binom{NI-KI}{LI-MI} * \binom{KI}{MI}$$

$$C(R) = \frac{\displaystyle\sum_{K_1=0}^{(1-R)N}\ \sum_{K_2=0}^{(1-R)N-K_1}\ \sum_{K_3=0}^{(1-R)N-K_1-K_2}\cdots\sum_{K_n=0}^{(1-R)N-\sum_{j=1}^{n-1}K_j}\ \prod_{i=1}^{n}\left[\binom{N_i-K_i}{L_i-M_i}*\binom{K_i}{M_i}\right]}{\displaystyle\sum_{K_1=0}^{N_1}\ \sum_{K_2=0}^{N_2}\ \sum_{K_3=0}^{N_3}\cdots\sum_{K_n=0}^{N_n}\ \prod_{i=1}^{n}\left[\binom{N_i-K_i}{L_i-M_i}*\binom{K_i}{M_i}\right]} \qquad (4).$$

$$C(R) = \frac{\displaystyle\sum_{K_1=M_1}^{(1-R)N}\left\{\binom{N_1-K_1}{L_1-M_1}\binom{K_1}{M_1}\sum_{K_2=M_2}^{(1-R)N-K_1}\left[\cdots\binom{N_{n-1}-K_{n-1}}{L_{n-1}-M_{n-1}}\binom{K_{n-1}}{M_{n-1}}\sum_{K_n=M_n}^{(1-R)N-\sum_{j=1}^{n-1}K_j}\binom{N_n-K_n}{L_n-M_n}\binom{K_n}{M_n}\right]\right\}}{\displaystyle\sum_{K_1=M_1}^{N_1-L_1+M_1}\left\{\binom{N_1-K_1}{L_1-M_1}\binom{K_1}{M_1}\sum_{K_2=M_2}^{N_2-L_2+M_2}\left[\cdots\binom{N_{n-1}-K_{n-1}}{L_{n-1}-K_{n-1}}\binom{K_{n-1}}{M_{n-1}}\sum_{K_n=M_n}^{N_n-L_n+M_n}\binom{N_n-K_n}{L_n-M_n}\binom{K_n}{M_n}\right]\right\}} \qquad (5).$$

and

$$LI-MI > NI-KI \quad \vee \quad MI > KI \implies B = 0 \quad .$$

To avoid overflow from large factorials, excessive computer time from automated cancellation of common factors, and error from approximations, it is helpful to employ logarithms in programming this function. One way of doing it is:

```
      COMMON ZERO,FL(1000)               )
      ZERO=0.                            )  at beginning of
      DO 10 I=1,1000                     )  main program
10    FL(I)=FL(I-1)+ALOG(FLOAT(I))       )
      :

      FUNCTION B(NI,KI,LI,MI)
      COMMON ZERO,FL(1000)
      IF (NI.LE.0.OR.NI.LT.LI.OR.LI.LT.MI) STOP
      ND=NI-KI
      LD=LI-MI
      IF (ND.LT.LD) GO TO 10
      B=EXP(FL(KI)-FL(MI)-FL(KI-MI)+FL(ND)-FL(LD)-FL(ND-LD))
      RETURN
10    B=0.
      RETURN
      END
```

With B available by such means as the foregoing one may write a function subroutine which returns C(R). An example is shown on the next page for the case in which n = 3.

```
      FUNCTION C(R)
      COMMON N1,L1,M1,N2,L2,M2,N3,L3,M3
      NB=(1.-R)*(N1+N2+N3)
      U=0.
      D=0.
      MA=M1+1
      NX=N1-L1+MA
      DO 30 KX=MA,NX
      K1=KX-1
      V=0.
      E=0.
      MB=M2+1
      NY=N2-L2+MB
      DO 20 KY=MB,NY
      K2=KY-1
      W=0.
      F=0.
      MC=M3+1
      NZ=N3-L3+MC
      DO 10 KZ=MC,NZ
      K3=KZ-1
      BT=B(N3,K3,L3,M3)
      IF (K1+K2+K3.LE.NB) W=W+BT
10    F=F+BT
      BT=B(N2,K2,L2,M2)
      IF (K1+K2.LE.NB) V=V+BT*W
20    E=E+BT*F
      BT=B(N1,K1,L1,M1)
      IF (K1.LE.NB) U=U+BT*V
30    D=D+BT*E
      C=U/D
      RETURN
      END
```

(This programming can be shortened if it is known beforehand that M_i will not be zero. Under some circumstances it can also be made more efficient if a table of C(R) vs R is desired, instead of C(R) for only a single value of R, by computing entries in the table first for small values of R and then just adding to the partial sum which is the numerator instead of recomputing the existing partial sum and the entire denominator.)

REFERENCE

1. Confidence and Reliability in a Finite Population", Probability and Statistics Note 1, 18 February 1971, by Chris Ashley.