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# Matrix Solution of the Helmholtz Equation Via Extended Separation of Variables

P. L. Overfelt
Physics Branch
Research and Technology Group
Naval Air Warfare Center Weapons Division, China Lake, CA 93555

**Abstract**

In this paper families of solutions to the two- and three-dimensional Helmholtz equation have been determined using the technique for solving partial differential equations known as extended separation of variables. In two dimensions this method results in three Nth order families of solutions composed of various combinations of polynomial and sinusoidal functions that have been cast into matrix form using a constituent matrices approach. In three dimensions standard separation of variables is used on the equations containing the coupled coordinates until vector-differential equations analogous to the two-dimensional case are obtained. An extended separation of variables solution to Maxwell's equations is found and subsequently shown to be the lowest order transverse electric mode of a uniform perfectly conducting rhombic waveguide.

## I. INTRODUCTION

The mathematical technique for solving partial differential equations known as separation of variables is so well known that it is seldom given much explanation. It has been mentioned in the by-now classic book by Miller [1] that "The method of separation of variables, .... although easy to illustrate for certain important examples, proves surprisingly subtle and difficult to describe in general." He states further that "... separation of variables is a method for finding solutions of a second-order partial differential equation in n variables by reduction of this equation to a system of n (at most) second-order ordinary differential equations."

The above statements with only slight modification can be applied also to the technique for solving PDEs known as *extended separation of variables* introduced by Gauchman and Rubel [2]. While separation of variables searches for solutions (of a PDE in the Cartesian coordinates x and y) of the form $u(x,y) = X_1(x)Y_1(y)$, i.e., a product of solutions of each independent variable alone, extended separation seeks solutions of the form $u(x,y)=\sum_{j=1}^{N} X_j(x)Y_j(y)$, i.e., sums of products of functions of x times functions of y. Obviously $N = 1$ reduces extended separation to the normal separation of variables technique.

In this paper we apply extended separation of variables to the Helmholtz equation in both two and three dimensions. We use a matrix polynomial formulation to solve the general N case in two dimensions. The three dimensional equation is then solved similarly but in much less detail. We then indicate the role the resulting types of solutions play in solving electromagnetic boundary value problems [3-7].

## II. MATRIX SOLUTION OF THE TWO-DIMENSIONAL HELMHOLTZ EQUATION

Assuming that we wish to solve

$$\left(\Delta + k_0^2\right) f(x,y,z) = 0 \tag{1}$$

in Cartesian coordinates and assuming that the z dependence is given by $e^{-ik_z z}$ while the time dependence is $e^{i\omega t}$, (1) becomes

$$\left(\Delta_t + k^2\right) u(x,y) = 0 \tag{2}$$

where $\Delta_t = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ , $k^2 = k_0^2 - k_z^2$, and $f(x,y,z) = u(x,y)e^{-i(k_z z - \omega t)}$ .

Let u(x,y) have the general form

$$u(x,y) = \sum_{j=1}^{N} X_j(x) Y_j(y) = X(x)^t\, Y(y) = Y(y)^t\, X(x) \tag{3}$$

where t denotes transpose and

$$X(x) = \begin{bmatrix} X_1(x) \\ \vdots \\ X_N(x) \end{bmatrix} \quad , \quad Y(y) = \begin{bmatrix} Y_1(y) \\ \vdots \\ Y_N(y) \end{bmatrix} \quad . \tag{4}$$

Substituting (3) and (4) into (2), the two-dimensional Helmholtz equation can be written in the form

$$X''(x)^t\, Y(y) + X(x)^t\, Y''(y) + k^2 X(x)^t\, Y(y) = 0 \tag{5}$$

where the primes signify differentiation with respect to the variable explicitly shown.

By Lemma 1.1 of Gauchman and Rubel [2], there exist real numbers, $y_1$, $y_2$, ... $y_N$ such that

$$\det\left[Y_i\left(y_j\right)\right]_{i,j=1,\ldots N} \neq 0 \quad . \tag{6}$$

Substituting $y_1$ ... $y_N$ into (5) successively, after some algebra one can obtain a system of N homogeneous linear equations given in matrix form as

$$SX''(x) + S''X(x) + k^2 SX(x) = 0 \tag{7}$$

where

$$S = \left[Y_i\left(y_j\right)\right]_{i,j=1,\ldots N} \quad , \tag{8a}$$

$$S'' = \left[Y_i''\left(y_j\right)\right]_{i,j=1,\ldots N} \quad , \tag{8b}$$

and

$$X''(x) = \begin{bmatrix} X_1''(x) \\ \vdots \\ X_N''(x) \end{bmatrix} \quad . \tag{8c}$$

Since S is given from (8a), from (6) its determinant is nonzero. Thus S has a unique inverse and (7) becomes

$$S^{-1}SX''(x)+\left[S^{-1}S''+k^2S^{-1}S\right]X(x)=0 \tag{9}$$

or

$$X''(x)=-\left[S^{-1}S''+k^2\mathbf{I}\right]X(x) \quad , \tag{10}$$

where **I** is the identity matrix. Setting $A = -S^{-1}S''$, (10) becomes

$$X''(x)=\left(A-k^2\mathbf{I}\right)X(x) \quad . \tag{11}$$

If we assume that the matrix A is an N x N general matrix (i.e., that it is nonsemisimple [8]), then A can be reduced to Jordan canonical form [2], or

$$A=\begin{bmatrix} \kappa^2 & 1 & 0 & 0\cdots & 0 \\ 0 & \kappa^2 & 1 & 0\cdots & 0 \\ 0 & 0 & \kappa^2 & 1\cdots & 0 \\ \vdots & & & \kappa^2 & \\ 0 & 0 & 0 & 0\cdots & \kappa^2 \end{bmatrix} \quad . \tag{12}$$

The constant, $\kappa$, is usually referred to as a separation constant when using the usual separation of variables technique. Thus the matrix A could be called a *separation matrix* in the extended formulation.

Substituting (12) into (11) and writing out (11) in component form, the resulting X(x) ordinary differential equations (ODEs) are

$$X_1''(x)+\left(k^2-\kappa^2\right)X_1(x)=X_2(x) \tag{13a}$$

$$X_2''(x)+\left(k^2-\kappa^2\right)X_2(x)=X_3(x) \tag{13b}$$

$$\vdots$$

$$X_N''(x)+\left(k^2-\kappa^2\right)X_N(x)=0 \quad . \tag{13c}$$

Equations (13) are individually solvable, beginning with $X_N(x)$ and successively solving for $X_{N-1}(x)$, $X_{N-2}(x)$, ... $X_1(x)$.

To determine the corresponding set of ordinary differential equations in y, one must return to (5) and substitute the transpose of eqn. (11). Then (5) becomes

$$X(x)^t\left\{\left(A-k^2\mathbf{I}\right)^t Y(y)+Y''(y)+k^2\mathbf{I}Y(y)\right\}=0 \tag{14}$$

or

$$Y''(y)=-A^t\,Y(y) \quad . \tag{15}$$

Thus the Y(y) ODEs in component form are

$$Y_1''(y)=-\kappa^2 Y_1(y) \tag{16a}$$

$$Y_2''(y)=-\kappa^2 Y_2(y)-Y_1(y) \tag{16b}$$

$$\vdots$$

$$Y_N''(y)=-\kappa^2 Y_N(y)-Y_{N-1}(y) \quad . \tag{16c}$$

Now the $Y_1(y)$ equation is immediately solvable while the remainder can be found successively from $Y_2(y)$ ... $Y_N(y)$ via, e.g., variation of parameters. For now it is assumed that $k^2$ and $\kappa^2$ are both real and that $k^2$ is positive while $\kappa^2$ could be positive, negative, or zero.

Considering (11) and (15), these two equations are vector-differential equations, and it is possible to reduce their degree by increasing their dimensionality [8]. For example to solve (15), one can define a new vector

$$M(y) = \begin{bmatrix} Y(y) \\ --- \\ Y'(y) \end{bmatrix} \tag{17}$$

such that

$$\frac{dM(y)}{dy} = DM(y) \tag{18}$$

where D is a 2N x 2N matrix given by

$$D = \left[\begin{array}{c|c} 0 & \mathbf{I} \\ \hline -A' & 0 \end{array}\right] . \tag{19}$$

The vector-differential equation in (18) has the formal general solution [8,9]

$$M(y) = e^{Dy} M(0) \tag{20}$$

where $e^{Dy}$ is the matrix exponential associated with D in (19), and M(0) is a vector containing the boundary conditions for both Y(y) and Y′(y) at y = 0 so that

$$M(0) = \left[\frac{Y(0)}{Y'(0)}\right] \quad . \tag{21}$$

While (20) is a very simple looking and elegant solution to (18), it is also a difficult form from which to determine explicit $Y_j(y)$ solutions. We have introduced (20) because when the value of N is large, a matrix formulation is the most compact way to achieve the solutions $Y_1(y)$, ..., $Y_N(y)$. For very small values of N, it is easier to solve the equations beginning with $Y_1(y)$ and progressing to $Y_2(y)$, etc. via variation of parameters. But variation of parameters quickly becomes extremely tedious for N larger than three or four.

Using (20), the matrix exponential can be written in a constituent matrix formulation [10] so that in general

$$e^{Dy} = \sum_{k=1}^{s} \left(Z_{k1} + Z_{k2} y + \ldots + Z_{k,m_k} y^{m_k - 1}\right) e^{\lambda_k y} \tag{22}$$

where $Z_{kj}$ are constituent matrices for the matrix exponential, s is the number of different eigenvalues associated with the matrix D in (19), $\lambda_k$ are the values of the different eigenvalues, and $m_k$ is the multiplicity of each different eigenvalue. In (22)

$$Z_{kj} = \frac{1}{(j-1)!(m_k - j)!} \left[\frac{C(\lambda)}{\psi_k(\lambda)}\right]^{(m_k - j)} \Bigg|_{\lambda = \lambda_k} \tag{23}$$

where $C(\lambda)$ is the reduced adjoint matrix of $(\lambda I - D)$ and $\psi_k(\lambda)$ is

$$\psi_k(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{m_k}} \tag{24}$$

and $\psi(\lambda)$ is the minimal polynomial associated with $(\lambda\mathbf{I}\text{-D})$ [10].

As mentioned previously, there are three possibilities: $\kappa^2 > 0$, $\kappa^2 < 0$, $\kappa^2 = 0$.

Case I: $\kappa^2 = 0$

When $\kappa^2 = 0$, the matrix D has the form in eqn. (19) with

$$A' = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} . \tag{25}$$

Thus

$$\lambda\mathbf{I} - D = \left[\begin{array}{c|c} \lambda\mathbf{I} & -\mathbf{I} \\ \hline A' & \lambda\mathbf{I} \end{array}\right] . \tag{26}$$

The characteristic equation of (26) is

$$\Delta(\lambda) = \lambda^{2N} . \tag{27}$$

Thus the only root in this case is $\lambda_1 = 0$ with a multiplicity, $m_1 = 2N$. Using the general constituent matrix formula given in (22) with $s = 1$, $m_1 = 2N$, and $\lambda_1 = 0$,

$$e^{Dy} = Z_{11} + Z_{12}y + \ldots + Z_{1,2N}y^{2N-1} \quad . \tag{28}$$

The minimal polynomial of (26) is the same as the characteristic equation in (27), i.e., $\psi(\lambda) = \lambda^{2N}$. This causes $\psi_1(\lambda) = 1$ in this case when (24) is used. The formula [10]

$$\Psi(\lambda,\mu) = \frac{\psi(\mu) - \psi(\lambda)}{\mu - \lambda} \tag{29}$$

(where $\lambda$ and $\mu$ are both scalars) can be used to obtain the reduced adjoint matrix associated with (26), i.e.,

$$C(\lambda) = \Psi(\lambda \mathbf{I}, D) \quad . \tag{30}$$

Using $\psi(\lambda) = \lambda^{2N}$, (29) becomes

$$\Psi(\lambda,\mu) = \frac{\mu^{2N} - \lambda^{2N}}{\mu - \lambda} = \sum_{p=1}^{2N} \mu^{2N-p}\lambda^{p-1} \quad . \tag{31}$$

Substituting the matrix D for $\mu$ in (31),

$$C(\lambda) = \sum_{p=1}^{2N} D^{2n-p}\lambda^{p-1} \quad . \tag{32}$$

Since $\psi_1(\lambda)$ is a constant in this case, the constituent matrix formula can be rewritten as (k = 1 only)

$$Z_{1j} = \frac{1}{(j-1)!(2N-j)!} C(\lambda)^{(2N-j)} \Big|_{\lambda=0} \qquad (33)$$

and $C(\lambda)^{(2N-j)}\Big|_{\lambda=0}$ means the (2N-j)-th derivative of C(λ) with respect to λ, evaluated at λ = 0.

Using (32) the derivatives of C(λ) can be easily written down:

$$C^{(\ell)}(\lambda) = \sum_{p=1}^{2N} \left[ \prod_{j'=1}^{\ell} (p-j') \right] D^{2N-p} \lambda^{p-\ell-1} \quad . \qquad (34)$$

Evaluating (33) using (34), we find that

$$Z_{11} = D^0 = \mathbf{I} \; ; \quad Z_{12} = D \; ; \ldots Z_{1,2N} = \frac{D^{2N-1}}{(2N-1)!} \quad . \qquad (35)$$

Thus given D in eqn. (19) with $A^t$ as in (25),

$$e^{Dy} = \mathbf{I} + Dy + \frac{D^2 y^2}{2!} + \ldots + \frac{D^{2N-1} y^{2N-1}}{(2N-1)!} \qquad (36)$$

For this case, we find that $D^{2N} = 0$ as well as its higher powers. Evaluating the powers of D, $e^{Dy}$ can finally be written explicitly as

$$e^{Dy} = \left[ \begin{array}{c|c} \frac{dF}{dy} & F \\ \hline \frac{d^2F}{dy^2} & \frac{dF}{dy} \end{array} \right] \qquad (37)$$

where the matrix F has the general form

$$F = \begin{bmatrix} y & 0 & 0 & 0 & \cdots & 0 \\ -\frac{y^3}{3!} & y & 0 & 0 & \cdots & 0 \\ \frac{y^5}{5!} & -\frac{y^3}{3!} & y & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \\ (-1)^{i+1}\frac{y^{2N-1}}{(2N-1)!} & (-1)^{i}\frac{y^{2N-3}}{(2N-3)!} & \cdots & \cdots & \cdots & y \end{bmatrix} \tag{38}$$

where i is the row number, and $\frac{d}{dy}$ in (37) denotes matrix derivative. Each entry in F is a single term polynomial in y. The general form (37) is true not only when $\kappa^2 = 0$ but also for any value of $\kappa$.

Returning to (20), we can rewrite it as

$$\begin{bmatrix} Y(y) \\ \hline Y'(y) \end{bmatrix} = \left[\begin{array}{c|c} \frac{dF}{dy} & F \\ \hline \frac{d^2F}{dy^2} & \frac{dF}{dy} \end{array}\right] \begin{bmatrix} Y(0) \\ \hline Y'(0) \end{bmatrix} . \tag{39}$$

Y(y) can be determined from the top row of (39) as

$$Y(y) = \frac{dF}{dy} Y(0) + FY'(0) \quad . \tag{40}$$

It is unnecessary to obtain Y′(y) in this way (except as a check) since once Y(y) is known, Y′(y) can be computed directly. Thus explicit knowledge of the matrix F is all that is needed to determine the Y(y) solutions.

For general N,

$$Y_1(y) = Y_1(0) + Y_1'(0)y \tag{41a}$$

$$Y_2(y) = Y_2(0) + Y_2'(0)y - Y_1(0)\frac{y^2}{2!} - Y_1'(0)\frac{y^3}{3!} \tag{41b}$$

$\vdots$

$$Y_N(y) = \sum_{\ell=1}^{N} (-1)^{\ell-1}\left[Y_{N+1-\ell}(0)\frac{y^{2\ell-2}}{(2\ell-2)!} + Y'_{N+1-\ell}(0)\frac{y^{2\ell-1}}{(2\ell-1)!}\right] \quad . \tag{41c}$$

Note that when the separation constant, κ, is zero in the separation matrix, A', the Y(y) solutions are all pure polynomials. For the case of standard separation of variables where N = 1, the only Y(y) solutions are Y constant or Y linear.

Returning to (11) to solve for the X(x) solutions in the $\kappa^2 = 0$ case, (11) reduces to

$$X_1''(x) + k^2 X_1(x) = X_2(x) \tag{42a}$$

$\vdots$

$$X_N''(x) + k^2 X_N(x) = 0 \quad . \tag{42b}$$

In solving (42) or (11), the matrix D now takes on the form

$$D = \left[\begin{array}{c|c} 0 & \mathbf{I} \\ \hline A - k^2\mathbf{I} & 0 \end{array}\right] \tag{43}$$

(and since we are now solving for solutions that are functions of x alone)

$$\frac{dM(x)}{dx} = DM(x) \tag{44}$$

where now

$$M(x) = \left[\begin{array}{c} X(x) \\ \hline X'(x) \end{array}\right] \quad , \quad M(0) = \left[\begin{array}{c} X(0) \\ \hline X'(0) \end{array}\right] \tag{45}$$

and

$$M(x) = e^{Dx} M(0) \tag{46}$$

with D given by (43).

Although $\kappa^2$ is still zero in this case, $k^2$, the transverse wave number, is assumed to be real and positive. Using (43)

$$\lambda \mathbf{I} - D = \left[\begin{array}{c|c} \lambda \mathbf{I} & -\mathbf{I} \\ \hline k^2 \mathbf{I} - A & \lambda \mathbf{I} \end{array}\right] \tag{47}$$

and the associated characteristic equation for (47) is

$$\Delta(\lambda) = \left(\lambda^2 + k^2\right)^N = \sum_{m=0}^{N} \binom{N}{m} k^{2m} \lambda^{2(N-m)} \tag{48}$$

where $\binom{N}{m}$ is the standard binomial coefficient. For this case ($\kappa^2 = 0$, $k^2 > 0$), there are two distinct roots, $\lambda_1 = ik$ and $\lambda_2 = -ik$, each with multiplicity $m_1 = m_2 = N$. The general constituent matrix formula in (22) with $s = 2$, $m_1 = m_2 = N$, $\lambda_1 = ik$, and $\lambda_2 = -ik$ becomes

$$e^{Dx} = \sum_{\ell=1}^{N} x^{\ell-1}\left[Z_{1,\ell}e^{ikx} + Z_{2,\ell}e^{-ikx}\right] \quad . \tag{49}$$

Since $\lambda_1$ and $\lambda_2$ are complex conjugates

$$Z_{2,\ell} = Z_{1,\ell}{}^{*} \quad . \tag{50}$$

Now the reduced adjoint matrix associated with (47) is

$$C(\lambda) = \sum_{m=0}^{N} \binom{N}{m} k^{2m} \sum_{p=1}^{2(N-m)} D^{2(N-m)-p} \lambda^{p-1} \quad . \tag{51}$$

Its derivatives are

$$C^{(\ell)}(\lambda) = \sum_{m=0}^{N} \binom{N}{m} k^{2m} \sum_{p=1}^{2(N-m)} \left[\prod_{q=1}^{\ell}(p-q)\right] D^{2(N-m)-p} \lambda^{p-\ell-1} \quad . \tag{52}$$

From eqn. (24), in this case

$$\psi_1(\lambda) = (\lambda + ik)^N \tag{53a}$$

and

$$\psi_2(\lambda) = \psi_1^*(\lambda) \quad . \tag{53b}$$

Since $\psi_1(\lambda)$ and $\psi_2(x)$ are now not constants but functions of $\lambda$, when (23) is used to obtain the constituent matrices, the higher-order derivatives are more difficult to obtain.

Via Leibnitz theorem,

$$\frac{d^\ell\left[C(\lambda)\psi_k^{-1}(\lambda)\right]}{d\lambda^\ell} = \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{d^{\ell-r}C(\lambda)}{d\lambda^{\ell-r}} \frac{d^r\left(\psi_k^{-1}\right)}{d\lambda^r} \tag{54}$$

where

$$\left[\psi_k^{-1}(\lambda)\right]^{(r)} = \frac{d^r\left(\psi_k^{-1}(\lambda)\right)}{d\lambda^r} = (-1)^r (N)_r (\lambda + ik)^{-(N+r)} \tag{55}$$

where $(N)_r$ is a Pochhammer polynomial [11].

Substitution of (54) and (55) into (23) provides a specific formula for the constituent matrices, i.e.,

$$Z_{1,j} = \frac{1}{(j-1)!(N-j)!} \left[ \sum_{r=0}^{N-j} (-1)^r \binom{N-j}{r} (N)_r \frac{C^{(N-j-r)}(\lambda)}{(2ik)^{N+r}} \right]_{\lambda=ik} \tag{56}$$

Using (51) and (52) in (56) and evaluating all derivatives at $\lambda = ik$,

$$Z_{1,j} = \frac{1}{(j-1)!(N-j)!}\left\{\sum_{r=0}^{N-j} \frac{(-1)^{N-j}}{2^{N+r}}(N)_r \right.$$

$$\left. \sum_{m=0}^{N} \binom{N}{m} \sum_{p=1}^{2(N-m)} i^{p-2N+j-1}(1-p)_{N-j-r} k^{2(m-N)+p+j-1} D^{2(N-m)-p} \right\}; \quad 1 \le j \le N \tag{57}$$

where again $(N)_r$ and $(1\text{-}p)_{N\text{-}j\text{-}r}$ are Pochhammer polynomials.

Substitution of (57) into (49) results in an explicit matrix form for $e^{Dx}$. Writing out the F matrix, for the case N = 2, F becomes

$$F = \begin{bmatrix} \frac{\sin kx}{k} & \frac{\sin kx}{2k^3} - \frac{x\cos kx}{2k^2} \\ 0 & \frac{\sin kx}{k} \end{bmatrix} \tag{58}$$

while for N = 3,

$$F = \begin{bmatrix} \frac{4\sin kx}{k} & \frac{2\sin kx}{k^3} - \frac{x\cos kx}{2k^2} & \frac{3\sin kx}{2k^5} - \frac{3x\cos kx}{8k^4} - \frac{x^2\sin kx}{8k^3} \\ 0 & \frac{4\sin kx}{k} & \frac{2\sin kx}{k^3} - \frac{x\cos kx}{2k^2} \\ 0 & 0 & \frac{4\sin kx}{k} \end{bmatrix} . \tag{59}$$

For general N, we have

$$F=\begin{bmatrix} [(N-1)!]^2\dfrac{\sin kx}{k} & \dfrac{[(N-1)!]^2}{2}\dfrac{\sin kx}{k^3}-\dfrac{[(N-2)!]^2}{2}\dfrac{x\cos kx}{k^2} & & \cdots \\ 0 & [(N-1)!]^2\dfrac{\sin kx}{k} & \ddots & \dfrac{[(N-1)!]}{2}\dfrac{\sin kx}{k^3}-\dfrac{[(N-2)!]^2}{2}\dfrac{x\cos kx}{k^2} \\ 0 & & \ddots & \\ \vdots & & & \ddots \\ \vdots & & & \\ 0 & \cdots & \cdots & [(N-1)!]^2\dfrac{\sin kx}{k} \end{bmatrix} \tag{60}$$

Thus in general F is an upper triangular matrix (in this case and for the X(x) solutions) with identical entries along the main diagonal and all succeeding semi-diagonals. Since the X(x) solutions are cumbersome to write out in component form for general N, we write out the N = 2 case simply for illustrative purposes.

Using (53) and (46),

$$X(x)=\frac{dF}{dx}X(0)+F\,X'(0) \tag{61}$$

or

$$X_1(x)=\left[X_1(0)-X_2'(0)\frac{x}{2k^2}\right]\cos kx+\left[\frac{X_1'(0)}{k}+X_2(0)\frac{x}{2k}+\frac{X_2'(0)}{2k^3}\right]\sin kx \tag{61a}$$

and

$$X_2(x)=X_2(0)\cos kx+X_2'(0)\frac{\sin kx}{k}\quad . \tag{61b}$$

Thus using (41a), (41b), and (61), for the N = 2 case a solution to the Helmholtz equation is

$$u(x,y) = [Y_1(0) + Y_1'(0)y]\left\{\left[X_1(0) - X_2'(0)\frac{x}{2k^2}\right]\cos kx\right.$$

$$\left. + \left[\frac{X_1'(0)}{k} + \frac{X_2'(0)}{2k^3} + X_2(0)\frac{x}{2k}\right]\sin kx\right\} \qquad (62)$$

$$+\left[Y_2(0) + Y_2'(0)y - Y_1(0)\frac{y^2}{2!} - Y_1'(0)\frac{y^3}{3!}\right]\left[X_2(0)\cos kx + X_2'(0)\frac{\sin kx}{k}\right]$$

Equation (62) is one example of a family (since N can be any positive integer in general) of increasingly complicated solutions to the Helmholtz equation where the solutions in y are pure polynomials multiplied by functions of x that are combinations of polynomials and sinusoids. Since the Helmholtz equation is symmetric with respect to x and y, by interchanging the roles of x and y in (62), (as well as X and Y) a second family of solutions can be generated, characterized by pure polynomials in x multiplied by combinations of polynomials and sinusoids in y.

Case 2: $k^2 > 0$, $0 \le \kappa^2 \le k^2$.

In this case for the Y(y) solutions, $\lambda$I-D still has the general form in (32) but $A^t$ is now

$$A^t = \begin{bmatrix} \kappa^2 & 0 & 0 & \cdots & 0 \\ 1 & \kappa^2 & 0 & & \\ 0 & 1 & \kappa^2 & & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & & \kappa^2 \end{bmatrix} \qquad (63)$$

and thus the characteristic equation associated with $\lambda$I-D is

$$\Delta(\lambda) = \left(\lambda^2 + \kappa^2\right)^N = \sum_{m=0}^{N} \binom{N}{m} \kappa^{2m} \lambda^{2(N-m)} \tag{64}$$

which is identical to (48) with κ replacing k. Thus

$$e^{Dy} = \sum_{\ell=1}^{N} y^{\ell-1}\left[Z_{1,\ell} e^{i\kappa y} + Z_{1,\ell} e^{-i\kappa y}\right] \ , \tag{65}$$

following the form of (49). We can obtain the constituent matrices using eqns. (51)-(57) with κ replacing k. Thus for the N = 2 case the F matrix is

$$F = \begin{bmatrix} \dfrac{\sin \kappa y}{\kappa} & 0 \\ -\left(\dfrac{\sin \kappa y}{2\kappa^3} - \dfrac{y \cos \kappa y}{2\kappa^2}\right) & \dfrac{\sin \kappa y}{\kappa} \end{bmatrix} . \tag{66}$$

Thus

$$Y(y) = \frac{dF}{dy} Y(0) + F Y'(0) \tag{67}$$

just as previously so that

$$Y_1(y) = Y_1(0) \cos \kappa y + Y_1'(0) \frac{\sin \kappa y}{\kappa} \ , \tag{68a}$$

$$Y_2(y) = -Y_1(0)\frac{y\sin\kappa y}{2\kappa^3} + Y_2(0)\cos\kappa y$$

$$-Y_1'(0)\left[\frac{\sin\kappa y}{2\kappa^3} - \frac{y\cos\kappa y}{2\kappa^2}\right] + Y_2'(0)\frac{\sin\kappa y}{\kappa} \quad . \tag{68b}$$

Similarly the X(x) solutions are found from

$$\lambda\mathbf{I} - D = \left[\begin{array}{c|c} \lambda\mathbf{I} & -\mathbf{I} \\ \hline -\left(A - k^2\mathbf{I}\right) & \lambda\mathbf{I} \end{array}\right] \tag{69}$$

where the characteristic equation is now

$$\Delta(\lambda) = \left(\lambda^2 + \left(k^2 - \kappa^2\right)\right)^N = \left(\lambda^2 + \delta^2\right)^N \tag{70}$$

with $\delta^2 = k^2 - \kappa^2$.

Immediately

$$e^{Dx} = \sum_{\ell=1}^{N} x^{\ell-1}\left[Z_{1,\ell}e^{i\delta x} + Z_{1,\ell}^{*}e^{-i\delta x}\right] \quad . \tag{71}$$

Again, we obtain the constituent matrices by using eqns. (51)-(57) with $\delta$ replacing k. Thus for the N = 2 case, the F matrix is

$$F = \begin{bmatrix} \dfrac{\sin\delta x}{\delta} & \dfrac{\sin\delta x}{2\delta^3} - \dfrac{x\cos\delta x}{2\delta^2} \\ 0 & \dfrac{\sin\delta x}{\delta} \end{bmatrix} \tag{72}$$

and thus

$$X_1(x) = \left[X_1(0) - X_2'(0)\frac{x}{2\delta^2}\right]\cos\delta x + \left[\frac{X_1'(0)}{\delta} + \frac{X_2(0)x}{2\delta} + \frac{X_2'(0)}{2\delta^3}\right]\sin\delta x \ , \tag{73a}$$

$$X_2(x) = X_2(0)\cos\delta x + X_2'(0)\frac{\sin\delta x}{\delta} \ . \tag{73b}$$

Thus for $\kappa^2 > 0$, $\delta^2 = k^2 - \kappa^2$, for N = 2 a solution to the Helmholtz equation is

$$\begin{aligned} u(x,y) = &\left\{\left[X_1(0) - X_2'(0)\frac{x}{2\delta^2}\right]\cos\delta x + \left[\frac{X_1'(0)}{\delta} + X_2(0)\frac{x}{2\delta} + \frac{X_2'(0)}{2\delta^3}\right]\sin\delta x\right\} \\ &\left\{Y_1(0)\cos\kappa y + Y_1'(0)\frac{\sin\kappa y}{\kappa}\right\} + \left\{X_2(0)\cos\delta x + X_2'(0)\frac{\sin\delta x}{\delta}\right\} \\ &\left\{\left[Y_2(0) + Y_1'(0)\frac{y}{2\kappa^2}\right]\cos\kappa y + \left[\frac{Y_2'(0)}{\kappa} - \frac{Y_1(0)y}{2\kappa} - \frac{Y_1'(0)}{2\kappa^3}\right]\sin\kappa y\right\} \end{aligned} \tag{74}$$

where normally $\delta$ would be $k_x$, $\kappa$ would be $k_y$, i.e., there are two nonzero separation constants in this case. When both separation constants are nonzero, the x and y solutions are both combinations of polynomials and sinusoids multiplied together. This shows that there is a third family of solutions in this case for general N.

In considering the cases where $\kappa^2 \geq 0$, we have so far been concerned only with the so-called propagating waves. However, when $\kappa^2 < 0$, this changes.

Case 3: $\kappa^2 < 0$ and real.

In this case

$$\lambda \mathbf{I} - D = \left[\begin{array}{c|c} \lambda \mathbf{I} & -\mathbf{I} \\ \hline A' & \lambda \mathbf{I} \end{array}\right] \tag{75}$$

where

$$A' = \begin{bmatrix} -\kappa^2 & 0 & 0 & \cdots & 0 \\ 1 & -\kappa^2 & 0 & & \vdots \\ 0 & 1 & -\kappa^2 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & -\kappa^2 \end{bmatrix} \tag{76}$$

(assuming that $\kappa^2 > 0$ so that we simply put a minus sign in front to make it negative) and so the characteristic equation associated with $\lambda$I - D is

$$\Delta(\lambda) = \sum_{m=0}^{N} \binom{N}{m} (-1)^m \kappa^{2m} \lambda^{2(N-m)} \tag{77}$$

The roots are now $\lambda_1 = \kappa$, $\lambda_2 = -\kappa$, each with multiplicity N and thus

$$e^{Dy} = \sum_{\ell=1}^{N} y^{\ell-1}\left(Z_{1,\ell}e^{\kappa y} + Z_{2,\ell}e^{-\kappa y}\right) \quad . \tag{78}$$

$C(\lambda)$ and $C^{(\ell)}(\lambda)$ are given by (51) and (52) with $k^2$ replaced by $-\kappa^2$. From (24) in this case

$$\psi_1(\lambda) = (\lambda+\kappa)^N \tag{79a}$$

but

$$\psi_2(\lambda) = (\lambda-\kappa)^N \quad . \tag{79b}$$

For this case, $Z_{1,\ell}$ and $Z_{2,\ell}$ will have to be computed separately. Without detail writing out the F matrix for the case N = 2,

$$F = \begin{bmatrix} \dfrac{\sinh \kappa y}{\kappa} & 0 \\ \dfrac{\sinh \kappa y}{2\kappa^3} - \dfrac{y\cosh \kappa y}{2\kappa^2} & \dfrac{\sinh \kappa y}{\kappa} \end{bmatrix} . \tag{80}$$

Thus for $\kappa^2 < 0$,

$$Y_1(y) = Y_1(0)\cosh \kappa y + Y_1'(0)\frac{\sinh \kappa y}{\kappa} \quad , \tag{81a}$$

$$Y_2(y) = -Y_1(0)y\frac{\sinh \kappa y}{2\kappa} + Y_2(0)\cosh \kappa y + Y_1'(0)\left[\frac{\sinh \kappa y}{2\kappa^3} - y\frac{\cosh \kappa y}{2\kappa^2}\right] + Y_2'(0)\frac{\sinh \kappa y}{\kappa} \tag{81b}$$

These represent the attenuating or evanescent wave solutions of the Helmholtz equation. The corresponding X(x) solutions for $\kappa^2 < 0$ are found similarly.

## III. MATRIX SOLUTION OF THE THREE-DIMENSIONAL HELMHOLTZ EQUATION

To solve

$$(\Delta + k_0^2)u(x,y,z) = 0 \tag{82}$$

using extended separation of variables, we assume a solution with the general form

$$u(x,y,z) = \sum_{j=1}^{N} X_j(x)Y_j(y)\zeta_j(z) \tag{83}$$

or

$$u(x,y,z) = \zeta(z)^t V(x,y) \tag{84}$$

where

$$\zeta(z) = \begin{bmatrix} \zeta_1(z) \\ \vdots \\ \zeta_N(z) \end{bmatrix} \quad , \quad V(x,y) = \begin{bmatrix} X_1(x)Y_1(y) \\ \vdots \\ X_N(x)Y_N(y) \end{bmatrix} \quad . \tag{85}$$

Using (84) and (85), (82) can be rewritten as

$$\zeta''(z)^t V(x,y) + \zeta(z)^t \Delta_t V(x,y) + k_0^2 \zeta(z)^t V(x,y) = 0 \quad . \tag{86}$$

Following the same procedure as for the two-dimensional case, we obtain two equations

$$\Delta_t V(x,y) + \left(k_0^2 \mathbf{I} - A\right) V(x,y) = 0 \tag{87a}$$

and

$$\zeta''(z) + A'\zeta(z) = 0 \quad . \tag{87b}$$

Equation (87b) is solved just as in the two-dimensional case with A given by (12). Equation (87a) can be solved as follows. Using A in (12), (87a) can be rewritten as

$$X_N''(x)Y_N(y) + Y_N''(y)X_N(x) + \left(k_0^2 - \kappa^2\right)X_N(x)Y_N(x) = 0 \tag{88a}$$

and

$$X_j''(x)Y_j(y) + Y_j''(y)X_j(x) + \left(k_0^2 - \kappa^2\right)X_j(x)Y_j(y) = X_{j+1}(x)Y_{j+1}(y); \quad j = N-1, \ldots 1 \quad . \tag{88b}$$

The Nth equation admits a standard separation of variables solution,

$$X_N(x) = A_0 \sin \alpha x + B_0 \cos \alpha x \tag{89a}$$

and

$$Y_N(y) = A_0' \sin \beta y + B_0' \cos \beta y \tag{89b}$$

where $\beta = \sqrt{k_0^2 - \kappa^2 - \alpha^2}$. But the jth equation given by (88b) provides the condition that

$$\frac{\partial}{\partial x}\left(\frac{X_{j+1}(x)}{X_j(x)}\right)\frac{\partial}{\partial y}\left(\frac{Y_{j+1}(y)}{Y_j(y)}\right) = 0 \quad , \tag{90}$$

thus either one derivative or the other in (90) must be zero. We are free to choose (arbitrarily) either one to be zero.

Choosing $\frac{\partial}{\partial y}\left(\frac{Y_{j+1}(y)}{Y_j(y)}\right) = 0$ implies that

$$Y_{j+1}(y) = cY_j(y) \tag{91}$$

where c is some constant which can be set to one without loss of generality. Thus once $Y_N$ is known, we know all of the $Y_j(y)$ solutions because of (91). Thus using (91), (88b) becomes

$$\frac{Y_j''(y)}{Y_j(y)} + \left(k_0^2 - \kappa^2\right) = \frac{X_{j+1}(x)}{X_j(x)} - \frac{X_j''(x)}{X_j(x)} \tag{92}$$

which is now separable in the standard sense. Setting both sides of (92) equal to the separation constant, $\alpha^2$ (to remain consistent with (89)), the subsequent equations are

$$Y_j''(y) + \left(k_0^2 - \kappa^2 - \alpha^2\right)Y_j(y) = 0 \tag{93a}$$

and

$$X_j''(x) + \alpha^2 X_j(x) = X_{j+1}(x) \quad . \tag{93b}$$

The solution to (93a) is given by (89b). After writing (93b) in component form with $1 \le j \le N-1$ and including the Nth equation (88a), the X(x) solutions may be found from the vector differential equation

$$X''(x) = -GX(x) \tag{94}$$

where

$$G = \begin{pmatrix} \alpha^2 & -1 & 0 & \cdots & 0 \\ 0 & \alpha^2 & -1 & \cdots & 0 \\ 0 & 0 & \alpha^2 & -1 & 0 \\ \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & \alpha^2 \end{pmatrix} . \tag{95}$$

Again since G is in Jordan canonical form, the techniques of Section II can be used to solve (94).

In (90) had we chosen $\frac{\partial}{\partial x}\left(\frac{X_{j+1}(x)}{X_j(x)}\right)$ to be zero, a similar equation to (94) would have been derived for the Y(y) solutions with $X_{j+1}(x) = X_j(x)$.

## IV. AN EXTENDED SEPARATION OF VARIABLES SOLUTION TO MAXWELL'S EQUATIONS

We would like to take the results of Section II and apply them to a particular problem. The solutions of Section II are particularly useful for solving electromagnetic waveguide problems where the waveguide is uniform along its length and is assumed to be perfectly conducting. Once solutions of the Helmholtz equation are known where

$$u(x,y) = E_z(x,y) \quad \text{or} \quad H_z(x,y) \tag{96}$$

the remaining electromagnetic field components are given by

$$H_x = \frac{i\omega\varepsilon_o\varepsilon}{k^2}\frac{\partial E_z}{\partial y} - \frac{ik_z}{k^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{i\omega\varepsilon_o\varepsilon}{k^2}\frac{\partial E_z}{\partial x} - \frac{ik_z}{k^2}\frac{\partial H_z}{\partial y}$$

$$E_x = \frac{-i\omega\mu_o\mu}{k^2}\frac{\partial H_z}{\partial y} - \frac{ik_z}{k^2}\frac{\partial E_z}{\partial x} \tag{97}$$

$$E_y = \frac{-i\omega\mu_o\mu}{k^2}\frac{\partial H_z}{\partial x} - \frac{ik_z}{k^2}\frac{\partial E_z}{\partial y} \quad .$$

As usual if u(x,y) = $E_z$ and $H_z$ = 0, transverse magnetic (TM) modes are obtained, while if u(x,y) = $H_z$ and $E_z$ = 0, transverse electric (TE) modes are obtained. The solution given in equation (62) composed of pure polynomials in y and products of polynomials and sinusoids in x is a solution of the two-dimensional Helmholtz equation that can also be used to obtain the lowest-order mode in a perfectly conducting waveguide with a rhombic cross section [4,6]. Setting

$$Y_1'(0) = X_2(0) = X_1(0) = Y_2(0) = Y_2'(0) = 0 \tag{98}$$

in (62), (62) becomes (for the lowest-order TE mode)

$$H_z(x,y) = Y_1(0)\left[X_1'(0)\frac{\sin kx}{k} + X_2'(0)\left(\frac{\sin kx}{2k^3} - \frac{x\cos kx}{2k^2}\right) - X_2'(0)\frac{y^2 \sin kx}{2k}\right] \quad . \tag{99}$$

Setting

$$a = \frac{X_1'(0)}{k} + \frac{X_2'(0)}{2k^3} \tag{100a}$$

$$b = \frac{-X_2'(0)}{2k} \tag{100b}$$

$$c = \frac{-X_2'(0)}{2k^2} \tag{100c}$$

to recapture the notation of Reference 4, (99) becomes

$$H_z(x,y) = \left(a + by^2 \sin kx + cx \cos kx\right) \tag{101}$$

with b - ck = 0. Equation (101) is an exact extended separation of variables solution to the Helmholtz equation and the numerical values of a, b, c, and k are used to satisfy the boundary conditions. The transverse field components (with $H_z$ as in (101) and $E_z = 0$) are

$$H_x(x,y) = \frac{-ik_z}{k^2}\left\{\left[c + k\left(a + by^2\right)\right]\cos kx - ckx \sin kx\right\} \quad , \tag{102a}$$

$$H_y(x,y) = \frac{-ik_z}{k^2} 2by \sin kx \quad , \tag{102b}$$

$$E_x = \frac{-i\omega\mu_0\mu}{k^2} 2by \sin kx \quad , \tag{102c}$$

$$E_y = \frac{i\omega\mu_0\mu}{k^2}\left\{\left[c + k\left(a + by^2\right)\right]\cos kx - ckx \sin kx\right\} \quad . \tag{102d}$$

Thus (101) and (102) are exact solutions to Maxwell's equations determined using extended separation of variables. With appropriate numerical values for the constants, these are the electromagnetic field components for the lowest-order mode of the perfectly conducting uniform rhombic waveguide where Maxwell's equations are satisfied exactly but the boundary conditions are satisfied numerically (and thus approximately) [4].

## V. CONCLUSIONS

Families of solutions to the two- and three-dimensional Helmholtz equation have been achieved using the technique for solving partial differential equations known as extended separation of variables. In two dimensions this method results in three Nth order families of solutions that have been cast into matrix form using a constituent matrices approach. In three dimensions standard separation of variables is used on the equations containing the coupled coordinates until vector differential equations analogous to the two-dimensional case are obtained. An extended separation of variables solution to Maxwell's equations is found and subsequently determined to be the lowest-order transverse electric mode of a perfectly conducting uniform rhombic waveguide.

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