Mathematics Note

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RATIONAL MODELING BY PENCIL-OF-FUNCTIONS METHOD

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ABSTRACT

Pole-zero modeling of signals, such as an electromagnetic-scatterer response, is considered in this paper. It is shown by use of pencil-of-functions theorem that (a) the true parameters can be recovered in the ideal case (where the signal is the impulse response of a rational function H(z)), and (b) the parameters are optimal in the functional dependence sense when the observed data are corrupted by additive noise or by systematic error. Although the computations are more involved than in all-pole modeling, they are considerably less than those required in iterative schemes of pole-zero modeling. The advantages of the method are demonstrated by a simulation example and through application to the electromagnetic response of a scatterer.

The paper also includes very recent and promising results on a new approach to noise correction. In contradistinction with spectral subtraction techniques, where only amplitude information is emphasized (and phase is ignored), we propose a method that (a) estimates the noise spectral density for the data frame, and then (b) performs the subtraction of the noise correlation matrix from the Gram matrix, of the signal.

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I. INTRODUCTION

Signal representation and approximation [1]-[4] is basic to (a) time-domain extraction of singularities of a scatterer's field pattern [5],[6],[16] and to (b) recursive digital-filter synthesis [7], [12]. It is also useful in (c) bandwidth compression of signals [2], and (d) time-domain measurement and testing of networks/channels. In the past few years, the problem has been researched extensively and a large body of literature has evolved. Notable success has been achieved in all-pole modeling where both the least-squares and the maximum-entropy formulations lead to the well known Yule-Walker normal equations. Although these equations are widely used, it must be remarked that in the presence of additive noise they are not entirely satisfactory in that they can lead to unacceptable bias and variance in the pole estimates. Even less satisfactory appears to be the situation for pole-zero modeling. First, let us point out the motivation for pole-zero models. Their need arises because frequently the underlying phenomenon warrants the use of both poles and zeros. This, for example, is the case in modeling the transient response of an electromagnetic scatterer, or the pulse response of an electronic network or channel. Their use is also being increasing recognized as a means for improvement in the quality of speech coders. On the other hand, pole-zero models are sometimes preferred because of their intrinsic efficiency, e.g., in the synthesis of digital filters with arbitrary frequency characteristics. For more discussion the reader is referred to Steiglitz [17], Cadzow [12], and Makhoul [1].

The classical approaches to pole-zero modeling are Pade' approximants [19], and the Prony method [18]. In their original forms both methods use a number of equations equaling the number of unknowns, although the more recent versions use a larger number of equations. In either case, the solution is noniterative, but, unfortunately, is sensitive to data noise. The true least-squares formulation of the problem, on the other hand, results in nonlinear equations which can only be solved by iterative methods, e.g., the Newton method [1], or the prefiltering method [17]. These algorithmic methods are not only plagued with convergence problems but yield a 'local' solution rather than a 'global' one. Therefore, interest in alternative formulations and solutions to the problem of pole-zero modeling continues to persist as evidenced by recent papers by Cadzow [12], Beex [20], Kumaresan [2], Henderson [22], Gueguen [23], and others. This paper discusses a unified approach to representing or approximating a given empirical signal x(t) by sum of exponentials, i.e., for finding the right hand side of

$$x(t) \cong y(t) = \sum_{i=1}^{n} d_i \, e^{s_i t} \leftrightarrow Y(s) \tag{1a}$$

$$Y(s) = \sum_{i=1}^{n} \frac{d_i}{s - s_i} \tag{1b}$$

or, equivalently, the right hand side of the sampled version

$$x(k) \simeq y(k) = \sum_{i=1}^{n} R_i (z_i)^k \leftrightarrow Y(z) \tag{2a}$$

$$\begin{aligned} Y(z) &= \sum_{i=1}^{n} \frac{R_i}{(1-z_i z^{-1})} \\ &= \frac{b_o + b_1 z^{-1} + \ldots + b_{n-1} z^{-n+1}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} \\ &= \frac{B(z)}{A(z)} \end{aligned} \tag{2b}$$

The poles s_i (or z_i in z-domain) are either real, or they occur in complex conjugate pairs.

When equality holds in (2a), the sampled signal x(k) is said to be rational of order n, and thus rationally representable. Additionally, if Re $s_i < 0$ (or, $|z_i| < 1$) it is said to be stable-rational of order n.

In the method described here, the given signal is processed in reverse-time by a cascade of first order digital filters to yield a family of information signals. If the cascade consists of n filters, the resulting n + 1 information signals will be found to be linearly dependent over the index set $I_\infty = \{0,1,..,\infty\}$, while any subset of these signals is linearly independent. This, in fact, is the reason for employing reverse-time filtering, although it must be remarked that in practice one must employ a finite index set $I_K = \{0,1,..,K-1\}$ for recording the signal x(k) as well as in the modeling computations. The Gram matrix F of these information signals is shown to contain the essential information on the denominator parameters of Y(z). Specifically, it is shown that A(z) is determined as

$$A(z) = (qz)^{-n} \left[\sum_{i=1}^{n+1} \sqrt{D_i}(qz-1)^{n+1-i} \right] / \sqrt{D_1}$$

where D_i are the diagonal cofactors of the matrix F. The numerator parameters are then determined using a least-squares fit, i.e., $\underline{b} = -P^{-1}\underline{c}$, where P and $\underline{c}$ are defined in the paper.

The entire procedure is thus noniterative and computationally efficient. Iterative methods, such as the modified Newton method [12], require as many as a hundred iterations, each involving a matrix inversion. Our computations are roughly equivalent to two matrix inversions. It is an extension of the method developed in [8] to reverse-time processing by first order filters. This formulation results in a lower order matrix (n+1 dimensional) than did the formulation in [8] (2n+1 dimensional). Examples presented demonstrate (i) noiseworthiness in the representation problem when data are corrupted by noise and (ii) the effectiveness of the method in the approximation problem. Comparison of the method with the maximum entropy method (or all-pole linear predictor) and the Prony method [1], [4] is also included in the paper.

An important feature of the paper is a new approach to noise correction. In contradistinction to spectral subtraction techniques, where only amplitude information is emphasized (and phase is ignored), we present a procedure that (a) estimates the noise spectral density for the data frame, and then (b) performs the subtraction of the noise correlation matrix from the Gram matrix of the noisy signal.

II. FIRST-ORDER FILTER BASED INFORMATION SIGNALS

In this and the next section we assume that $K = \infty$. From a practical standpoint it is only necessary that a finite K be selected such that $x(k) \simeq 0$ for $k \geq K$ (so that use of the upper limit ∞ instead of K-1 on summations may be permitted). We define the reverse-time first-order filtered signals as (see Fig. 1)

$$
\begin{aligned}
x_1(k) &= x(k) \\
x_2(k) &= q x_2(k+1) + x_1(k) \\
&\ \ \vdots \\
x_N(k) &= q x_N(k+1) + x_n(k)
\end{aligned}
\tag{3}
$$

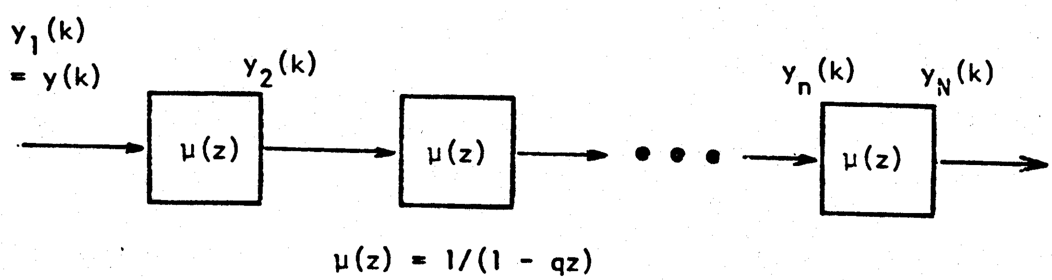

Fig. 1. Reverse-time processing by first-order filters

where $N = n+1$, and $x_i(K) = 0$ for $i = 1, 2, \ldots, N$. Further, $0 < q < 1$.

This family of signals[1], which we shall call information signals, possesses the interesting property stated below.

Lemma 1

If $x(k) = y(k)$ is stable-rational of order n with poles z_i, then the corresponding information signals are also stable rational of order n with the poles z_i:

$$y_{i+1}(k) = \sum_{\ell=1}^{n} \frac{R_\ell}{(1-q\, z_\ell)^i} (z_\ell)^k \tag{4}$$

Proof: We prove this by induction. For i=0 the statement is trivially true since (4) is identical to (2) for this case. Assuming it to be true for i-1, let us proceed to prove it is true for i.

From (3)

$$y_{i+1}(k) = q y_{i+1}(k+1) + y_i(k)$$

which is readily shown to be equivalent to

$$y_{i+1}(k) = \sum_{\nu=k}^{\infty} q^{\nu-k}\, y_i(\nu)$$

$$= \sum_{\ell=1}^{n} \frac{R_i}{(1-q\, z_\ell)^{i-1}} z_\ell^{\,k} \sum_{\nu=k}^{\infty} q^{\nu-k} (z_\ell)^{\nu-k}$$

(from induction hypothesis)

The result of equation (3) follows immediately by observing that the last summation equals $1/(1-q\, z_\ell)$.

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[1] The nomenclature 'information signal' is not to be confused with traditional information theoretic concepts. It is used here because these signals will be shown to yield the denominator parameters of Y(z).

An alternative proof of the lemma follows by observing that the response of the anti-causal filter $\mu(z) = 1/(1-qz)$ to the input $z_\ell{}^k$ is $\mu(z_\ell)\, z_\ell{}^k$.

Before leaving this section, we remark that the set $y_1, \ldots, y_n$ is linearly independent, while the set $y_1, \ldots, y_n, y_{n+1}$ is linearly dependent.

III. DETERMINATION OF PARAMETERS VIA PENCIL-OF-FUNCTIONS THEOREM FOR RATIONALLY REPRESENTABLE SIGNALS

In this section we will determine the signal parameters for the case $x(k) = y(k)$, i.e., where the signal is rationally representable. We will call z_ℓ (see (2)) the poles of the impulse response, R_ℓ the corresponding residues, and $\zeta_\ell = \{(z_\ell)^k\}$ the associated modes. Note that the poles occur in conjugate pairs whenever complex, as do the residues, since y is real.

The significance of Lemma 1 of the previous section arises from the fact that each of the information signals contains the modes $\zeta_\ell = \{(z_\ell)^k\}$, $\ell = 1, \ldots, n$. Further, the pencil-of-signals[2] $\gamma\, y_{i+1} + y_i$ also contains all these modes unless γ equals one of the poles; in the latter case, i.e., when $\gamma = z_m$, $\gamma\, y_{i+1} + y_i$ does not contain the mode $\zeta_m = \{(z_m)^k\}$. This results in the following observation

Lemma 2. The set

$$(qz_m - 1)y_2 + y_1,\ (qz_m - 1)y_3 + y_2,\ \ldots,\ (qz_m - 1)y_N + y_n \qquad (5)$$

is linearly dependent for $m = 1, 2, \ldots, n$ where z_m are the poles of the right hand side of (2).

<u>Definition</u>. Define the N x N dimensional Gram matrix (recall, $N = n+1$)[11]

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[2] The terminology pencil-of-functions is derived from literature in physics and mathematics; see, for example, Gantmacher [9] where $A + \lambda B$ is called a pencil of matrices A and B parametrized by sealer λ. See also Gueguen[10] for recent usage.

$$F = \begin{bmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_N \rangle \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \langle y_N, y_1 \rangle & \cdots & \langle y_N, y_N \rangle \end{bmatrix},$$

$$\langle y_i, y_j \rangle = \sum_{k=0}^{K-1} y_i(k) y_j(k) \qquad (6)$$

or, equivalently

$$F = \sum_{k=0}^{K-1} \underline{f}(k) \underline{f}^T(k) \qquad (7)$$

where

$$\underline{f}^T(k) = [y_1(k)\; y_2(k)\; \ldots\; y_N(k)].$$

We can now apply the pencil-of-functions theorem of reference [8] to obtain the central theoretical result of this section. A statement of pencil-of-functions theorem is given in Appendix A.

Theorem 1. The poles of the impulse response y(k) must satisfy the equation

$$\sum_{i=1}^{N} \sqrt{D_i}\, (qz-1)^{N-i} = 0 \qquad (8)$$

where D_i are signed square-roots of the diagonal cofactors of Gram matrix F.

The proof of the theorem follows immediately upon application of the pencil-of-functions theorem to the set (5). The signs of the square-roots are taken to be the signs of the cofactors of the first row of F (see also Appendix A). Now, the denominator of the model is given by

$$A(z) = D_1^{-1/2} (qz)^{-n} \sum_{i=1}^{N} \sqrt{D_i}\, (qz-1)^{N-i} \qquad (9)$$

This follows from (8) by dividing through by z^n and by normalizing the coefficients so that the leading coefficient becomes unity.

The numerator parameters can be found by the method of least squares, specifically by solving the linear equation

$$P\,\underline{b} = \underline{c} \tag{10}$$

where $\underline{b} = [b_o\ b_1\ \ldots\ b_{n-1}]^T$, $\underline{c} = [c_o\ c_1\ \ldots\ c_{n-1}]^T$, $P = [p_{ij}]$; and

$$p_{ij} = \langle u_i, u_j\rangle \tag{11a}$$

$$c_i = \langle u_i, y\rangle \tag{11b}$$

Here u_i denotes the impulse response of $z^{-i}/A(z)$. Note that $u_i(k) = u(k-i)$ where $u(k)$ is the impulse response (i.e., inverse z-transform) of $1/A(z)$. All inner products are summed from $k = 0$ to $K-1$.

Remarks. Before leaving this section we remark that the parameters characterizing the signal, i.e., the coefficients of the polynomials $A(z)$ and $B(z)$, are recovered exactly. It is assumed of course that the signal is of the form (1) and that the true model order is known.

The idea of reverse-time integration was proposed by Carr in [13] and Jain in [14]. Here, we have generalized the concept of reverse-time processing to the case of first-order filter processing. Note that the first order filter $1/(1-qz)$, used above, encompasses integration; just let $q = 1$.

IV. EFFECT OF MISSING TAIL

In Sections II and III it was assumed that K was large enough such that $x(k) = 0$ for $k \geq K$. We now consider the effect of choice of K which does not meet this requirement, i.e., the effect of a missing tail. Unfortunately, this does not lend itself to a tractable analytical study for the general nth order, or even a general second order case. Therefore, we will present analytical results for a first order signal, and some experimental results for a second order signal. As may be expected, the estimated poles may not coincide with the poles of the signal on $[0, \infty)$. Thus even for the rational signals the estimated poles will not coincide with the true poles.

First Order Signal -

Consider the signal $x(k) = a^k$ for $k = 0,1,..,K-1$ where $0 < a < 1$. Then

$$x_2(k) = \frac{1-(aq)^{K-k}}{1 - aq} a^k$$

and the diagonal cofactors of the Gram matrix F are

$$D_1 = \frac{1}{(1-aq)^2} \left\{ \frac{1-a^{2K}}{1-a^2} - 2 \frac{(aq)^K - a^{2K}}{1 - a/q} + \frac{(aq)^{2K} - a^{2K}}{1 - q^{-2}} \right\}$$

$$\simeq \frac{1}{(1-aq)^2} \left\{ \frac{1-a^{2K}}{1-a^2} - 2 \frac{a^{2K}}{1-a/q} + \frac{a^{2K}}{q^{-2}-1} \right\}$$

$$D_2 = \frac{1-a^{2K}}{1-a^2}$$

The approximation for D_1 holds if $q^K \ll 1$. The second and third terms on the right-hand side of D_1 are produced by the missing tail of the signal. Table 1 summarizes the results on a specific case, namely for $a = 0.9$; the value of q was 0.6.

Table 1

Effect of Record Length K

First order Signal: True Pole = 0.9

K	100	50	20
Estimated Pole $\hat{a}$	0.90000	0.89999	0.89622

Second Order Signal -

Consider the signal $x(k) = 0.9^k \sin(0.2k)$. Note that the true poles $0.88206 \pm j0.17880$. Results obtained for various record lengths are given in Table 2; the value of q was taken to be 0.6.

Table 2

Effect of Record Length K
Second Order Signal

K	100	50	20
Estimated pole pair	0.88206± j0.17880	0.88204± j0.17879	0.86862± j0.17598

V. MODELING IN THE PRESENCE OF NOISE

In Section III we modeled a noisefree signal from its samples. The effect of additive noise is now considered. The samples available are

$$x(k) = y(k) + \sigma\, w(k) \tag{12}$$

where $w(k)$ is a zero mean white noise process and σ is an unknown positive constant. As in Section II, the information signals are again generated by processing $x(k) = x_1(k)$ by the filter cascade of Fig. 1. (see Fig. 2).

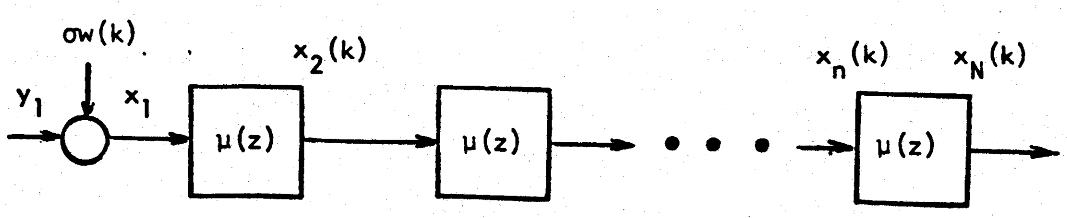

Fig. 2. Noisy signal through first-order filters

Note, because of the linearity of the filters, that

$$x_i(k) = y_i(k) + \sigma\, w_i(k) \tag{13}$$

Then it can be seen that the expected value of the Gram matrix G of the vector signal $\underline{x}(k) = [x_1(k)\ x_2(k)\ \ldots\ x_N(k)]^T$ is[3]

$$\mathrm{E}\,G = \mathrm{E} \sum_{k=0}^{K-1} \underline{x}(k)\, \underline{x}^T(k) = F + \sigma^2 W \tag{14}$$

where W is the covariance matrix of the unit noise vector sequence $\underline{w}(k) = [w_1(k)\ w_2(k)\ \ldots\ w_N(k)]^T$ and is known before hand (see Appendix B). To estimate σ^2 and F we use the following criterion.

<u>Jain's Identification Criterion</u> [15]

Consistent with the noise and bias models the estimated Gram matrix should achieve a minimum possible determinant.

Using the above criterion and equation (14), the following estimation procedure has been developed.

<u>Step 1</u>. Estimate σ^2 as

$$\hat{\sigma}^2 = \frac{1}{G^{-1} \odot W} \tag{15}$$

where $\odot$ denotes matrix inner product (i.e., $Z \odot W = \Sigma\Sigma\, z_{ij}\, w_{ij}$).

<u>Step 2</u>. $\hat{F} = G - \hat{\sigma}^2 W$ (16)

<u>Step 3</u>. Use $\hat{F}$ in estimating A(z) and B(z) via (9) and (10).

The justification for formulas (15) and (16) is given in Appendix B.

Remark - The minimization of the determinant of estimated F results in optimal functional dependence between the information signals. This determinant is a measure of the dependence of the set [φ]; a zero value, of course, implies perfect linear dependence of the set.

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[3] The expected value operator is denoted as E.

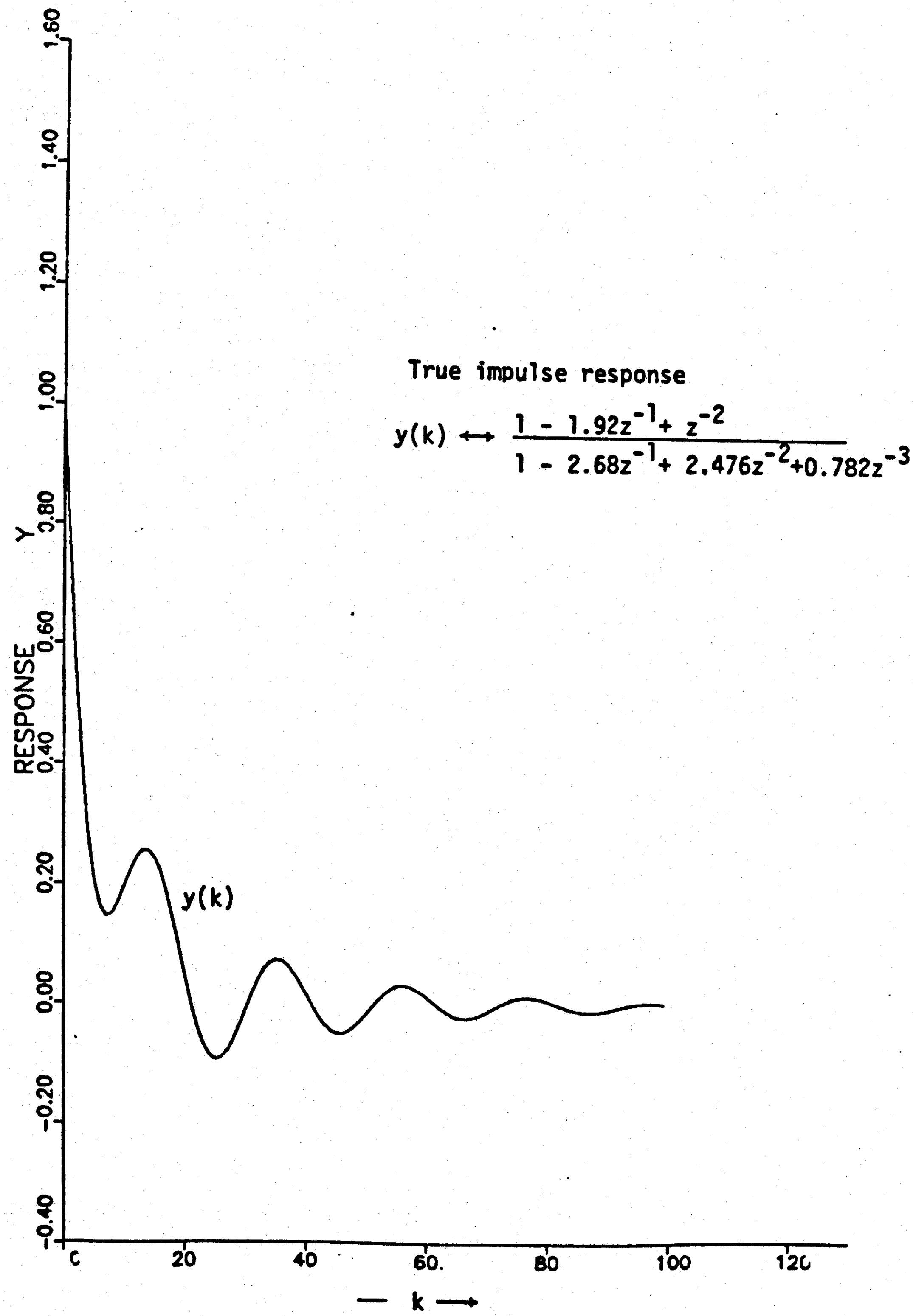

Fig. 3(a) True impulse response of a third order transfer function

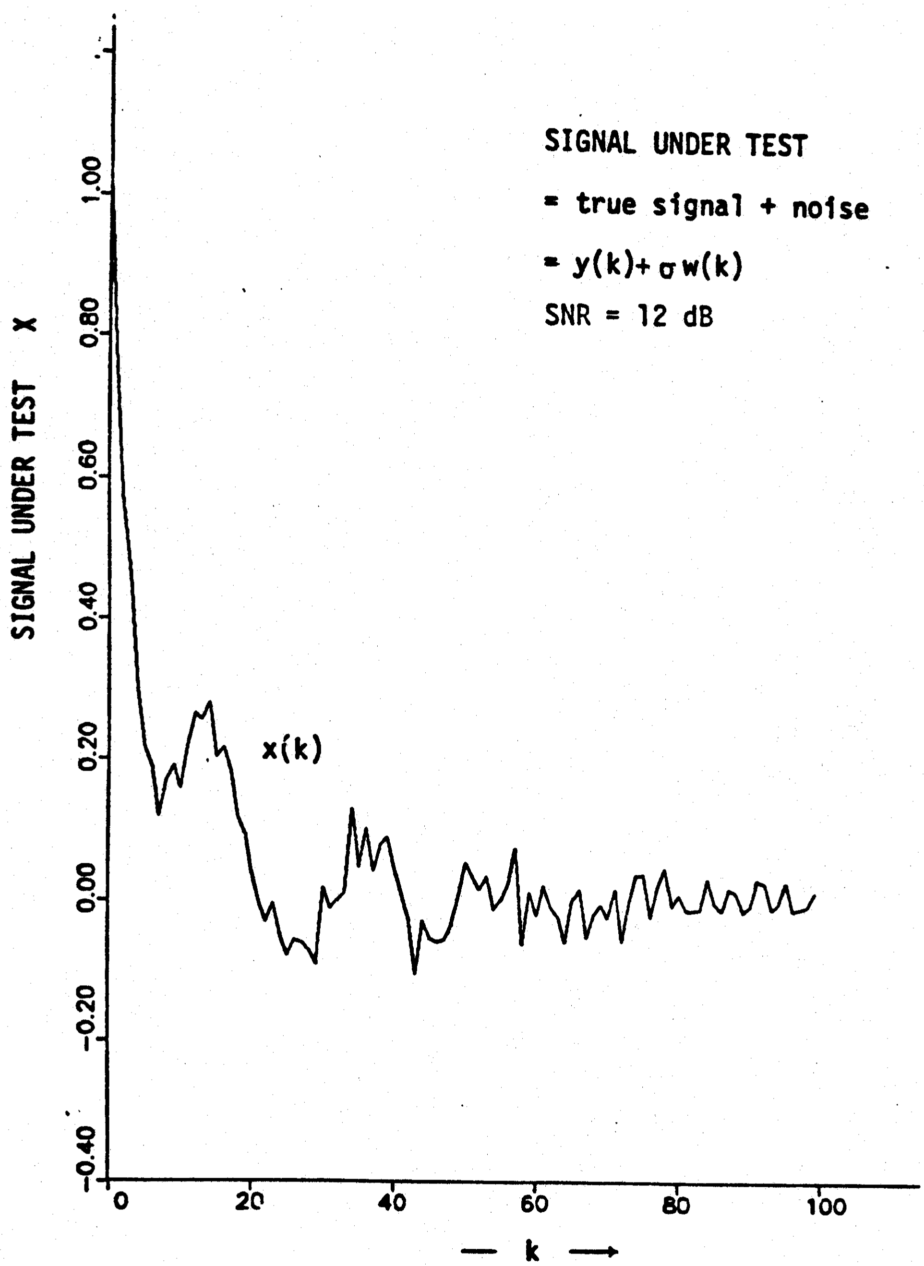

Fig.3(b) A simulated noisy signal under test

VI. A COMPUTER SIMULATION EXAMPLE

Let

$$y(k) \leftrightarrow \frac{1 - 1.92z^{-1} + z^{-2}}{1 - 2.68z^{-1} + 2.476z^{-2} - 0.782z^{-3}}$$

$$= \frac{(1 - e^{j\beta}z^{-1})(1 - e^{-j\beta}z^{-1})}{(1 - 0.8452z^{-1})(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \qquad (17)$$

$$(\beta = 0.28379;\ r = 0.96187,\ \theta = 0.30528)$$

be truncated at $k = 99$. The signal to be tested is formed as

$$x(k) = y(k) + \sigma\, w(k) \qquad (18)$$

where $w(k)$ is a zero mean, uncorrelated noise sequence. The positive scalar σ is chosen to be 0.0425 so that the signal-to-noise ratio is 12 dB. The true signal $y(k)$ and the noisy signal $x(k)$ are shown in Fig. 3.

The signal under test was modeled by

1. Pencil of functions method (without applying noise correction); reverse-time processing pole q was taken as 0.8.

2. Pencil-of-functions method with estimation of σ^2 and F; the value of q was again taken to be 0.8.

3. All-pole covariance[4] technique. Minimum error criterion, rather than equal energy criterion was used to establish the gain parameter.

4. Pole-zero model using Prony method. Note that the denominator parameters (and, of course, the poles) are the same as that for the all-pole covariance method [4]. The numerator parameters are then determined by least-squares fit.

Fifty simulation runs, each with a different sample of noise, were performed. The 'mean' and 'standard deviation (S.D.)' of the various quantities

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[4] The all-pole autocorrelation method yields very similar results in the present case; hence, the autocorrelation method is not included.

of interest are shown in Table 3. Of course, the model order was taken to be 3 and the number of signal samples used were 100.

A graphical portrayal of the z-domain poles for 10 runs is given in Figures 4 to 6. The location of the true poles is at the centers of the circles shown in these figures. Judging from the normalized mean square errors in Table 3, as well as from the scattergrams of the poles, it appears that the pencil-of-functions method can perform reliable modeling of a rational signal even when it is masked by noise. As is widely known, the Prony method (and of course the LPC covaraince method) perform quite poorly in the presence of noise.

It is sometimes claimed that the Prony method (or the all-pole-covariance method) performs well with short data-frame. We give the poles of ten runs with the first 10 data points of the noisy signal used in the above experiments. These poles are given in Fig. 7. Again a wide and unreliable scatter of the poles is produced.

VII. APPLICATION TO AN ELECTROMAGNETIC PULSE (EMP)

As a real world application we consider the use of pencil-of-functions method to the transient response of a conducting pipe tested at the ATHAMAS-I EMP simulator. The conducting pipe is 10m long and 1m in diameter. Hence, the true resonance of the pipe is expected to be in the neighborhood of 14MHz. Also, the pipe has been excited in such a way that it is reasonable to expect only odd harmonics at the scattered fields. The data measured are the integral of the E-field: i.e., the measured variable is a voltage. The transient response used for analysis is shown in Fig. 8 by the solid line. The results of analysis by the pencil-of-functions method are given in Table 4 for an 8th order model; the model response, with an error of 0.0125, is shown in Fig. 8

Table 3

Comparison of Pencil-of-Functions Method With

All-pole covariance and Prony Methods

(Results of fifty noisy runs: SNR=12 dB)

<table>
<tr><th rowspan="2">Mean ± S.D. of</th><th colspan="4">METHOD</th></tr>
<tr><th>POF</th><th>POF with noise correction</th><th>All-pole covariance</th><th>Prony</th></tr>
<tr><td>a_1</td><td>-2.5974 ± 0.0216</td><td>-2.6753 ± 0.0185</td><td colspan="2">-0.5200 ± 0.0747</td></tr>
<tr><td>a_2</td><td>2.3438 ± 0.0388</td><td>2.4698 ± 0.0331</td><td colspan="2">-0.2334 ± 0.1123</td></tr>
<tr><td>a_3</td><td>-0.7288 ± 0.0187</td><td>-0.7803 ± 0.0160</td><td colspan="2">0.0161 ± 0.0710</td></tr>
<tr><td>Error (NMSE)</td><td>0.0314 ± 0.0095</td><td>0.0055 ± 0.0052</td><td>0.1726 ± 0.0164</td><td>0.1499±0.0098</td></tr>
</table>

Note: From (17) the true parameters are $a_1 = -2.68$, $a_2 = -2.476$ and $a_3 = -0.782$

The denominator parameters for the all-pole-covariance method are the same as those of the Prony method

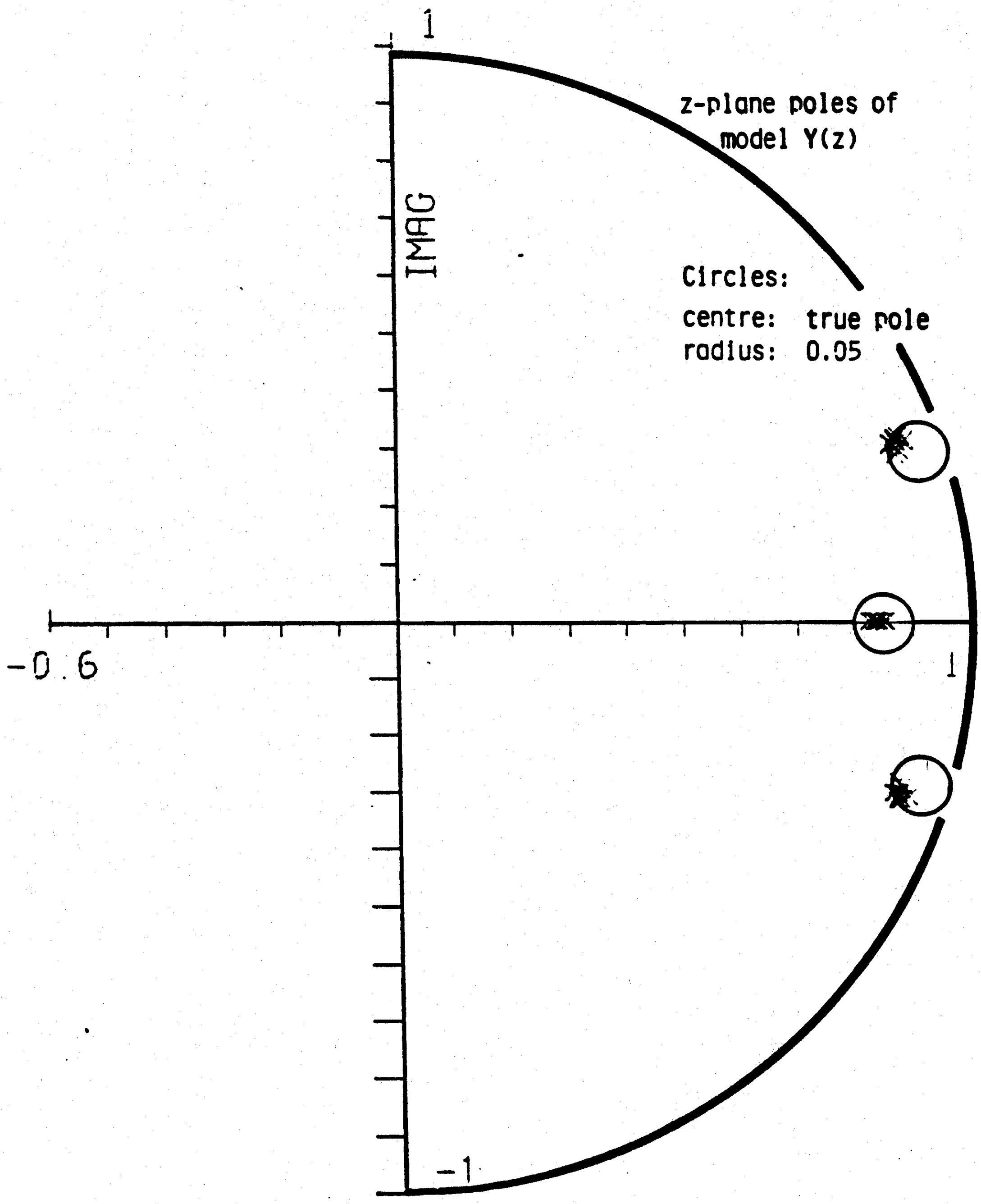

Fig.4. Poles obtained in ten (10) simulation runs by pencil-of-functions method.

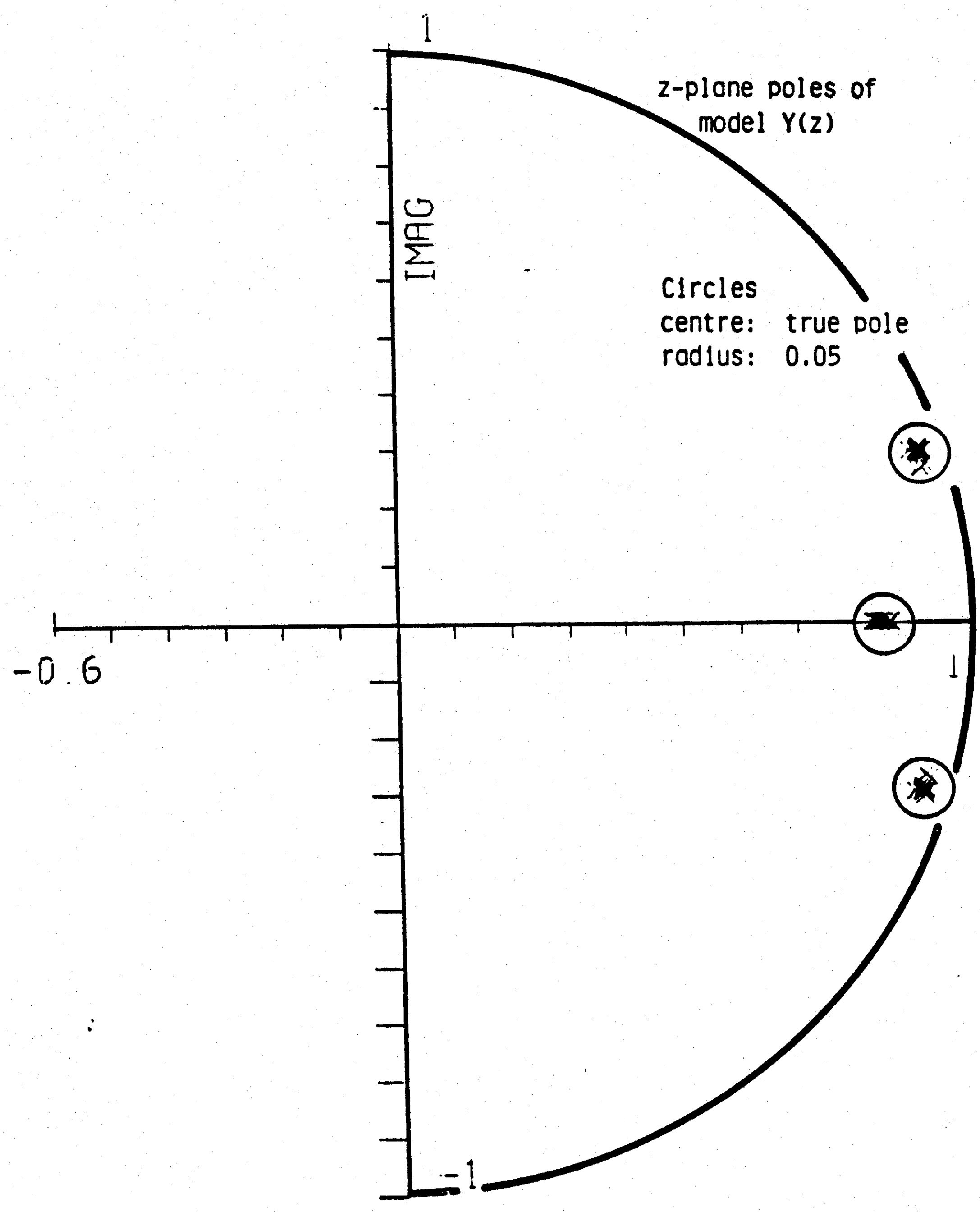

Fig.5. Poles obtained in ten (10) simulation runs by pencil-of-functions method with noise estimation.

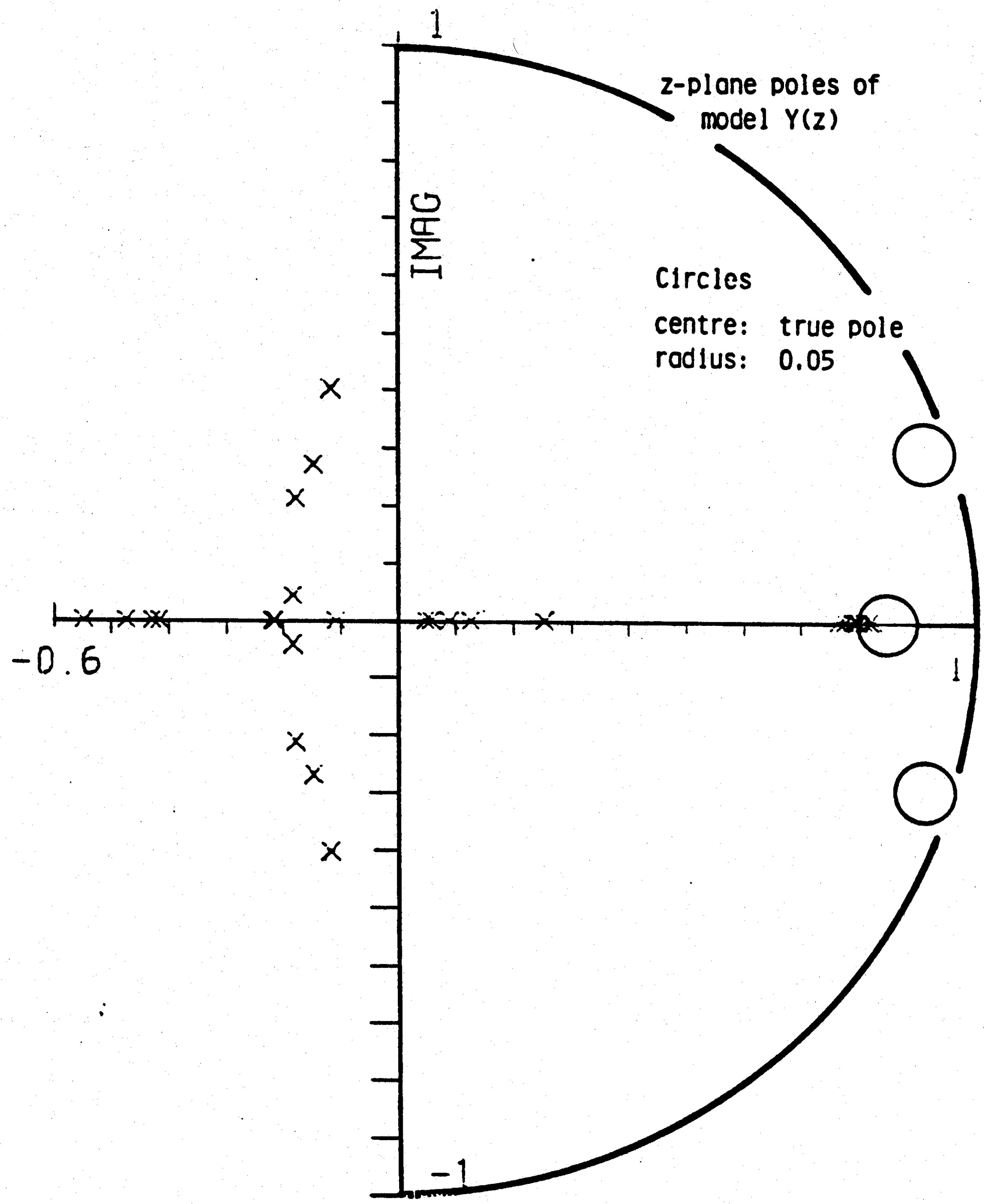

Fig.6. Poles obtained in ten (10) simulation runs by Prony method.

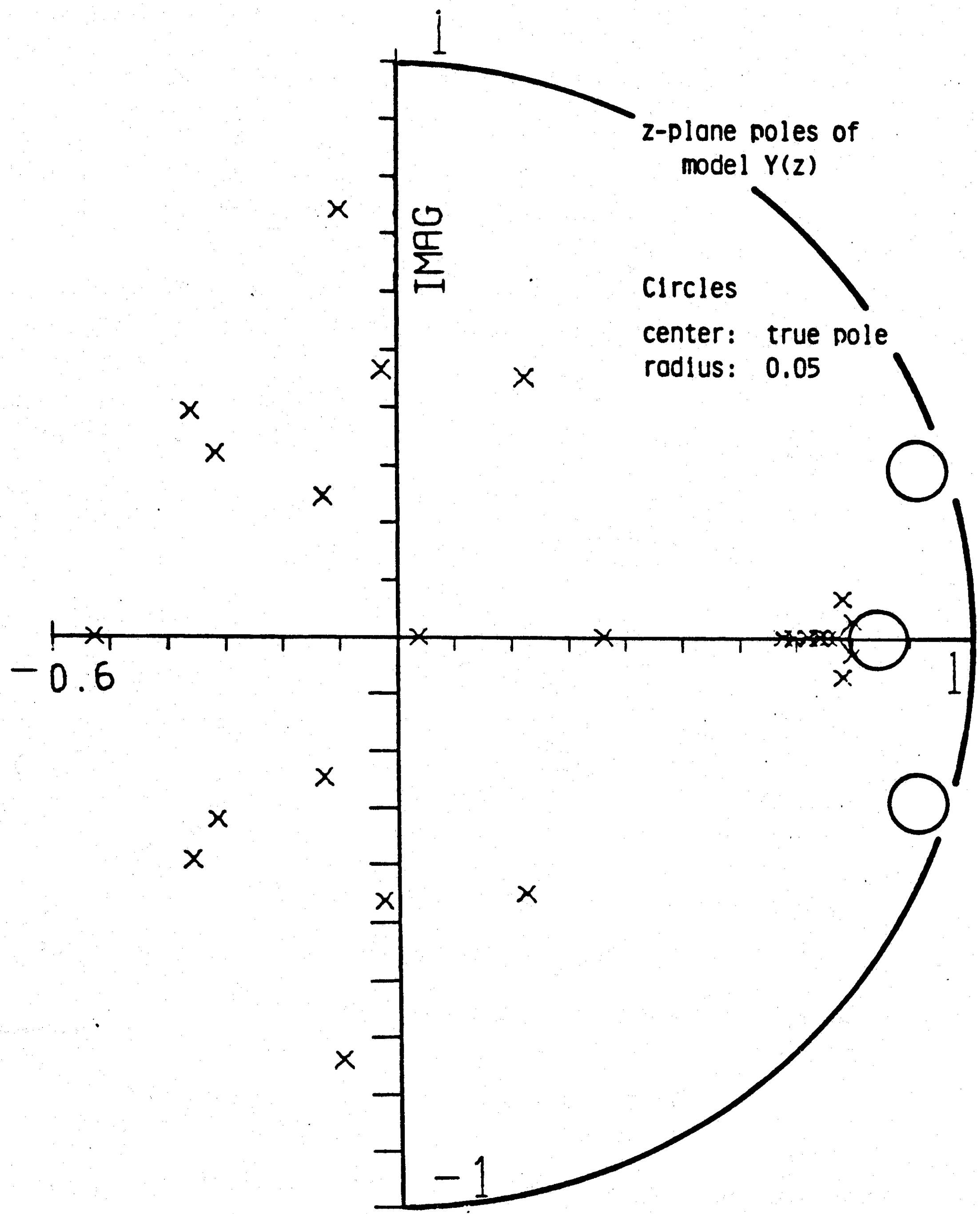

Fig. 7. Poles obtained in ten (10) simulation runs by Prony method with short frame (K=10).

by the dotted line. The sampling interval is $\Delta = 0.97656$ ns and the number of samples analyzed is $K = 245$. Note that noise estimation and correction, as described in Section V, has been used in the analysis.[4]

Table 4

Poles of a Scatterer Response Estimated by Pencil-of-Functions Method (q=0.8)

fundamental	$-5.72 \pm j\ 68.63$ Mrad/s	(10.96 MHz)
3rd harmonic	$-30.65 \pm j\ 212.60$ Mrad/s	(33.83 MHz)
curve-fit pair	$-1.95 \pm j\ 8.72$ Mrad/s	(1.42 MHz)
curve-fit pair	$-20.17 + j\ 95.58$ Mrad/s	(15.55 MHz)

VIII RECURSIVE DIGITAL FILTER SYNTHESIS EXAMPLE

As a final example we consider the use of pencil-of-functions technique to digital filter synthesis. Suppose the desired impulse response is [12]

$$h_d(k) = \begin{cases} 0.25, & k=0 \\ \mathrm{Sin}(0.25k)/k, & 1 \le k \le K = 256 \end{cases}$$

It represents the causal part of the inverse DFT of a low-pass filter with cutoff at 0.25 Hz. The application of the pencil-of-functions technique, with $q = 0.4$, yields the following filter

$$H(z) = \frac{0.25-0.38841z^{-1}+0.29346z^{-2}-0.05783z^{-3}-0.42025z^{-4}}{1 \quad -2.46160z^{-1}+2.78530z^{-2}-1.52226z^{-3}+0.34885z^{-4}}$$

with a normalized mean-square error 0.00473. Note that the minimum value of NMSE, as obtained in [12] by iterative methods,[5] is 0.00346.

[4]The computer output listings are given in Appendix D.

[5]Seventy iterations were needed to achieve the minimum value.

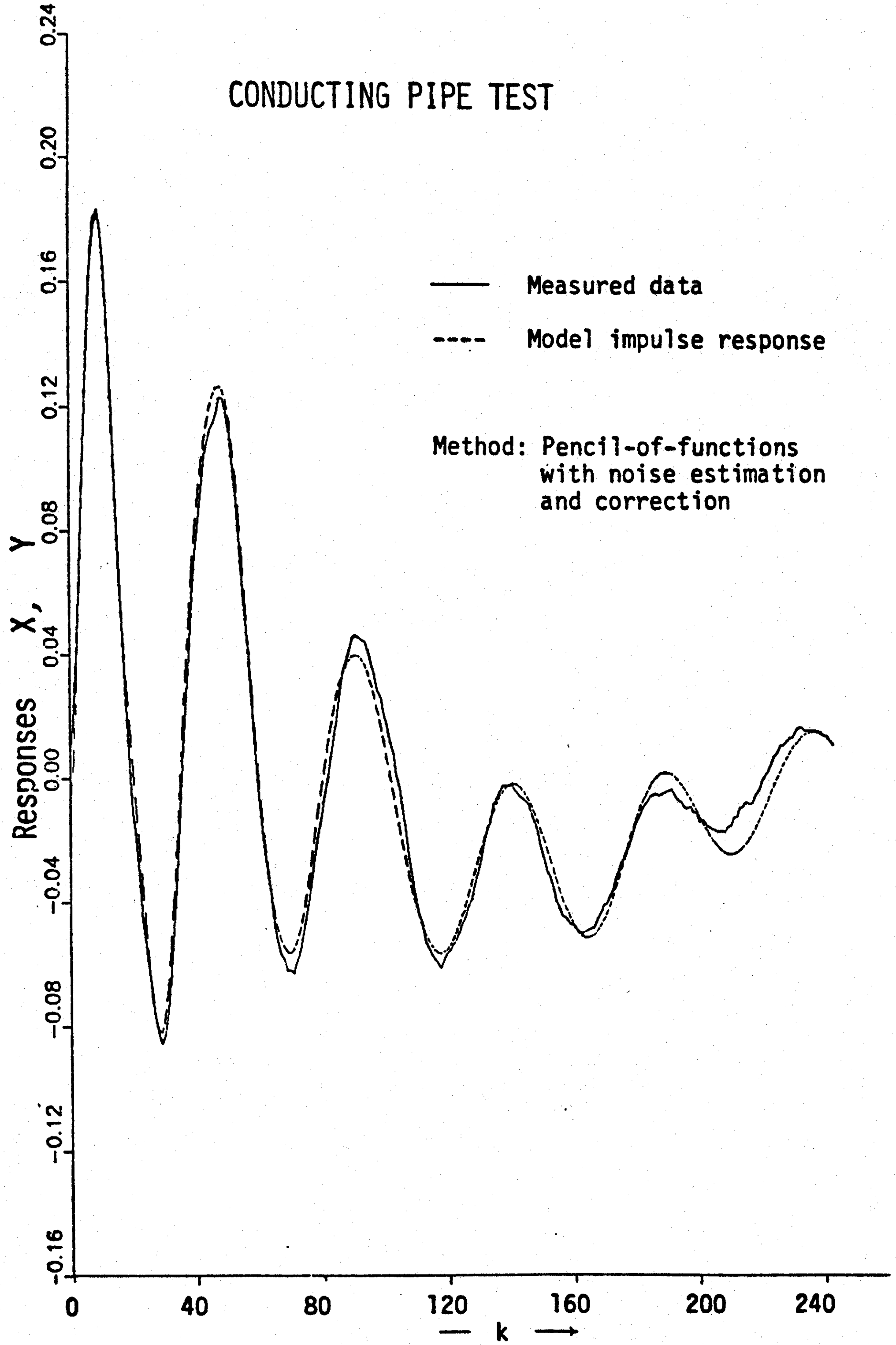

Fig. 8. Comparison of measured data (of the response of of a conducting pipe) and model impulse response

IX. CONCLUSIONS

Pole-zero modeling of signals has been considered in this paper. It was shown that for rational signals the true parameters can be recovered from the Gram matrix of the information signals. The latter were formed by reverse-time processing of the given signal by a cascade of first order digital filters. Further, we have formulated a new approach to noise estimation and correction by minimizing the determinant of the estimated Gram matrix. The examples demonstrate the practicality of the approach, not only because the computations are noniterative, but also because the poles of the signal are estimated quite accurately. It is felt that the method can be used in a broad range of applications, for example, finding the singularities of a scatter response, modeling of speech and in spectrum analysis.

Extension of the technique to modeling of multichannel signals with common modes (or singularities) is possible. This work is underway. A second area of extension pertains to the case where the filters $\mu(z)$ are chosen to be high-pass. This might be useful when the signal contains an undersirable low frequency drift component.

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APPENDIX A

PENCIL OF FUNCTIONS

A useful mathematical entity arises by combining two given functions defined on a common interval together with a scalar parameter

$$f(t,\gamma) = \gamma g(t) + h(t). \tag{A.1}$$

We call f a pencil of functions g(t) and h(t) parametrized by γ. To avoid obvious triviality, g(t) is not permitted to be a scalar multiple of h(t).

Our work requires consideration of sets of pencils

$$\gamma g_1(t) + h_1(t),\ \gamma g_2(t) + h_2(t),\ \dots, \gamma g_n(t) + h_n(t) \tag{A.2}$$

wherein the functions* $g_1(t),\dots,g_n(t)$ and $h_1(t),\dots,h_n(t)$ span separately a common n-dimensional subspace L_n in the function space. For a fixed set of values of parameters γ, the pencils obviously reduce to a set of functions, and the particular values chosen determine properties such as the linear dependence or independence of the set. The main result concerning the linear dependence of pencil sets is derived in [8] and can be stated as follows.

Theorem: Given that the pencil set (2) is linearly dependent, the parameters γ must satisfy the polynomial equation

$$\begin{aligned} &\gamma^n\sqrt{G}[g_1,g_2,\dots,g_n] \pm \gamma^{n-1}\Sigma\sqrt{G}[g_{i_1},\dots;h_{k_1},\dots,g_{i_{n-1}}] \\ &\pm \dots \pm \gamma\Sigma\sqrt{G}[h_{k_1},\dots;g_{i_1},\dots,h_{k_{n-1}}] \\ &\pm \sqrt{G}[h_1,h_2,\dots,h_n] = 0 \end{aligned} \tag{A.3}$$

In every sum term here, the i's and k's form a complete complementary set of indices over the integers 1,2,...,n; furthermore, the notation $G[f_1,\dots,f_n]$ stands for the determinant of the n-dimensional Gram matrix [11]

*All functions are defined on a common interval [a,b], with the usual inner product denoted as

$$\langle f,g\rangle = \int_a^b f(t)g^*(t)dt.$$

of the functions $f_1,\ldots,f_n$, i.e.,

$$G[f_1,\ldots,f_n] = \det[g_{ik} = \langle f_i,f_k\rangle], \quad i,k = 1,\ldots,n. \tag{A.4}$$

Lastly, we remark that the sign of each sum term is to be determined as indicated in [8].

The above discussion is equally valid for discrete-time signals. To this end the functions $f(t)$, $g_i(t)$ and $h_i(t)$ must be replaced by the sequences $f(k)$, $g_i(k)$ and $h_i(k)$, and of course the inner product must be redefined as

$$\langle f,g\rangle = \sum_{k=0}^{K-1} f(k)g^*(k) \tag{A.5}$$

APPENDIX B

NOISE ESTIMATION AND CORRECTION

We observed in Section V that

$$E\, G = F + \sigma^2 W \tag{B1}$$

where G is the Gram matrix of the noisy information signals $x_1,\ldots,x_{n+1}$. Clearly, a good estimator of F is

$$\hat{F} = G - \sigma^2 W \tag{B2}$$

Unfortunately, this estimator is not useable because σ^2 is unknown. We must estimate it using a property of the true information signals stated in Section III: The true information signals $y_1,\ldots,y_{n+1}$ are linearly dependent and their Gram matrix is singular. Thus we require

$$|\hat{F}| = |G - \hat{\sigma}^2 W| = 0 \tag{B3}$$

Assuming that $\hat{\sigma}^2$ is small, and retaining only the first two terms of the Taylor series, we have

$$|G| - \hat{\sigma}^2 \Sigma |(G,W)_i| = 0 \tag{B4}$$

where the notation $(G,W)_i$ stands for the matrix obtained by replacing the ith column of G by the ith column of W.

A little manipulation of (B4) now readily yields relation (16) of Section V.

Precomputation of W:

Recall that W is the covariance matrix of the vector sequence $\underline{w}(k) = [w_1(k) \;\; w_2(k) \; \ldots \; w_{n+1}(k)]^T$. By Wiener's theorem this matrix is exactly equal to the Gram matrix of the vector impulse response, $\underline{r}(k)$ of the filter cascade of Fig. 1. That is,

$$W = \sum_{k=0}^{K-1} \underline{r}(k)\, \underline{r}^T(k) \qquad \text{(B5)}$$

where $r_1(k) = \delta(k)$, and $r_{i+1}(k) = \mathcal{Z}^{-1}(\mu(z))^i$, $i=1,\ldots,n$. Obviously, this matrix is dependent only on the values of q, the cascade filter-pole, and the record length K. It can be computed before estimating $\hat{\sigma}^2$ and $\hat{F}$.