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Different Representations of Dyadic Green's Functions for a Rectangular Cavity

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Several different, but equivalent, expressions of the dyadic Green's functions for a rectangular cavity have been derived and tabulated in this report. The mathematical relations between the dyadic Green's function of the vector potential type and that of the electric type are shown in detail. This work supplements the one not fully treated by Morse and Feshbach.

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FIGURE

1. INTRODUCTION

The dyadic Green's function for a rectangular cavity has been studied by Morse and Feshbach.[1] The function that they introduced is of the vector potential type, hereby denoted by $\bar{\bar{G}}_A$, corresponding to the dyadic version of the vector Green's function for the vector Helmholtz equation. Two forms of $\bar{\bar{G}}_A$ were obtained by Morse and Feshbach. One form is complete, but the other is not. They mentioned that the two forms are equivalent when a longitudinal part is added to the incompleted form, but the exact relations were not discussed.

This report details the derivation of several alternative representations of the dyadic Green's functions of both the vector potential type and the electric type for a rectangular cavity. Although the two types of functions are intimately related, it is more direct to use the function of the electric type that would bypass the tedious differentiation of discontinuous series for the evaluation of the fields in a source region.

2. DYADIC GREEN'S FUNCTIONS OF THE VECTOR POTENTIAL TYPE AND OF THE ELECTRIC TYPE

The classification of dyadic Green's functions of various types and kinds has been discussed.[2,3] Here, it is sufficient to review two types of functions pertaining, respectively, to the vector potential function and the electric field. The dyadic Green's function of the vector potential type satisfied the differential equation

$$\nabla^2\bar{\bar{G}}_A + k^2\bar{\bar{G}}_A = -\bar{\bar{I}}\delta\left(\bar{R}-\bar{R}'\right) \tag{1}$$

where

$k^2 = \omega^2\mu_o\epsilon_o$

$\bar{\bar{I}}$ = the idem factor = $\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$

$\delta(\bar{R}-\bar{R}')$ = the three-dimensional delta function $= \delta(x-x')\,\delta(y-y')\,\delta\,(z-z')$.

A cap ($\wedge$) denotes a unit vector. The dyadic Green's function of the electric type satisfies the differential equation.

$$\nabla\times\nabla\times\bar{\bar{G}}_e + k^2\bar{\bar{G}}_e = \bar{\bar{I}}\delta\,\bar{R}-\bar{R}'\,. \tag{2}$$

The relation between $\bar{\bar{G}}_A$ and $\bar{\bar{G}}_e$ is

$$\left(\bar{\bar{G}}_e = \bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right)\cdot\bar{\bar{G}}_A\,. \tag{3}$$

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[1] *P.M. Morse and H. Feshback, Methods of Theoretical Physics, Part II, McGraw-Hill Book Company, New York (1953).*

[2] *C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, PA (1972).*

[3] *C.-T. Tai, Eigen-Function Expansion of Dyadic Green's Functions, Mathematics Note 28, Weapons Systems Laboratory, Kirtland Air Force Base, Albuquerque, NM (July 1973).*

For cavities, we are seeking the kind of functions that satisfy the boundary condition

$$\hat{n} \times \bar{\bar{G}}_e = 0 \tag{4}$$

or

$$\hat{n} \times \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla\right) \cdot \bar{\bar{G}}_A = 0. \tag{5}$$

Previously, we called these functions of the first kind and denoted them, respectively, by $\bar{\bar{G}}_{e1}$ and $\bar{\bar{G}}_{A1}$. Here, we omit the subscript "ℓ" for convenience. Later on the characteristics of the function of the second kind are briefly mentioned.

3. EIGEN-FUNCTION EXPANSION OF $\bar{\bar{G}}_A$ FOR A RECTANGULAR CAVITY

The rectangular cavity under consideration has the configuration shown in figure 1.

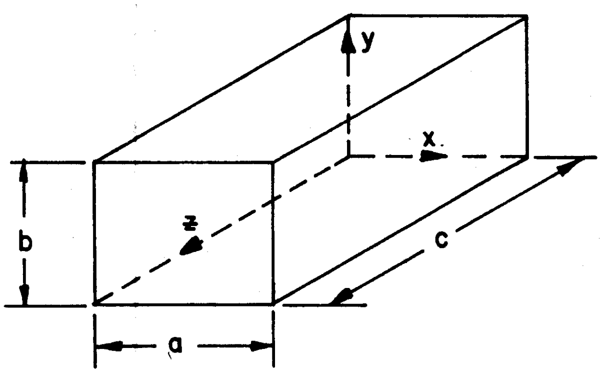

Figure 1. A Rectangular Cavity and the Designation of the Coordinate System.

Following the Ohm-Rayleigh method,[2,3] we expand first the singular function $\bar{\bar{I}}\delta(\bar{R} - \bar{R}')$ in terms of the vector wave functions $\bar{L}_{oo}$, $\bar{M}_{eo}$, and $\bar{N}_{oo}$ defined as follows:

$$\bar{L}_{oo} = \nabla\psi_{oo} \tag{6}$$

$$\bar{M}_{eo} = \nabla \times \left(\psi_{eo}\hat{z}\right) \tag{7}$$

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[2] *C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, PA (1972).*

[3] *C.-T. Tai, Eigen-Function Expansion of Dyadic Green's Functions, Mathematics Note 28, Weapons Systems Laboratory, Kirtland Air Force Base, Albuquerque, NM (July 1973).*

$$\bar{N}_{oe} = \frac{1}{K} \nabla \times \nabla \times \left(\psi_{oe} \hat{z}\right) \tag{8}$$

where

$$\psi_{oo} = \sin k_x x \sin k_y y \sin k_z z$$

$$\psi_{eo} = \cos k_x x \cos k_y y \sin k_z z$$

$$\psi_{oe} = \sin k_x x \sin k_y y \cos k_z z$$

$$k_x = \frac{m\pi}{a}, k_y = \frac{n\pi}{b}, k_z = \frac{\ell\pi}{c}$$

$$m, n, \ell = 0, 1, 2, \ldots$$

$$K^2 = k_x^2 + k_y^2 + k_z^2$$

The orthogonal properties of these vector wave functions result in

$$\bar{\bar{I}}\delta(\bar{R} - \bar{R}') = \sum_{\ell,m,n} C_{mn} \left[\frac{k_c^2}{K^2} \bar{L}_{oo} \bar{L}'_{oo} + \bar{M}_{eo} \bar{M}'_{eo} + \bar{N}_{oe} \bar{N}'_{oe}\right] \tag{9}$$

where the primed functions are defined with respect to the primed variables x', y', and z' pertaining to $\bar{R}'$, and

$$C_{mn} = \frac{4\left(2 - \delta_o\right)}{abc\, k_c^2}$$

$$k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\delta_o = \begin{cases} 1, & \ell \text{ or } m \text{ or } n = 0 \\ 0, & \ell, m, n \neq 0. \end{cases}$$

The constant k_c corresponds to the cutoff wave number of a rectangular waveguide with a cross section a x b. The derivation of equation (9) follows the same steps as described by Tai.[2,3] In view of equation (1),

$$\bar{\bar{G}}_A = \sum_{\ell,m,n} \frac{C_{mn}}{K^2 - k^2} \left[\frac{k_c^2}{K^2} \bar{L}_{oo} \bar{L}'_{oo} + \bar{M}_{eo} \bar{M}'_{eo} + \bar{N}_{oe} \bar{N}'_{oe}\right] \tag{10}$$

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[2] *C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, PA (1972).*

[3] *C.-T. Tai, Eigen-Function Expansion of Dyadic Green's Functions, Mathematics Note 28, Weapons Systems Laboratory, Kirtland Air Force Base, Albuquerque, NM (July 1973).*

An expression for $\bar{\bar{G}}_A$ without the $\bar{L}_{oo}\bar{L}'_{oo}$ terms was given by Morse and Feshbach's equation 13.3.46.[1] The series containing the $\bar{L}_{oo}\bar{L}'_{oo}$ terms not only is responsible for the field in a source region, but also contributes to the electric field in a source-free region. The expression for $\bar{\bar{G}}_A$ as given by equation (10) can be written in a different form with the modal functions commonly used in a waveguide theory, particularly, by Felsen, Marcuvitz, and their followers.[4] These modal functions have been used also by Morse and Feshbach. They are defined by

$$\bar{\ell}_o = \phi_o \hat{z} \tag{11}$$

$$\bar{m}_e = \nabla_t \phi_e x \hat{z} \tag{12}$$

$$\bar{n}_o = \nabla_t \phi_o \tag{13}$$

where

$$\phi_o = \sin k_x x \sin k_y y$$

$$\phi_e = \cos k_x x \cos k_y y$$

$$k_x = \frac{m\pi}{a}, \; k_y = \frac{n\pi}{b}, \; m, n = 0, 1, \ldots$$

The vector wave functions used in expanding $\bar{\bar{G}}_A$ are related to these modal functions. Thus, it is not difficult to show

$$\bar{L}_{oo} = \bar{n}_o \sin k_z z + k_z \bar{\ell}_o \cos k_z z$$

$$\bar{M}_{eo} = \bar{m}_e \sin k_z z$$

$$\bar{N}_{eo} = \frac{1}{K}\left(- k_z \bar{n}_o \sin k_z z + k_c^2 \bar{\ell}_o \cos k_z z\right).$$

In terms of $\bar{\ell}_o$, $\bar{m}_e$, and $\bar{n}_o$, equation (10) can have the form

$$\bar{\bar{G}}_A = \sum_{\ell, m, n} \frac{C_{mn}}{K^2 - k^2} \left[k_c^2 \bar{\ell}_o \bar{\ell}'_o \cos k_z z \cos k_z z' + \left(\bar{m}_e \bar{m}'_e + \bar{n}_o \bar{n}'_o\right) \sin k_z z \sin k_z z' \right]. \tag{14}$$

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[1] *P.M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill Book Company, New York (1953).*

[4] *L.B. Felsen and N. Marcuvitz,. Radiation and Scattering of Waves, Prentice Hall, Inc., Englewood Cliffs, N.J. (1973).*

Now, the sum over the index ℓ can be evaluated in a closed form by use of the relations[5]

$$\sum_{\ell=1} \frac{1}{K^2 - k^2} \sin k_z z \sin k_z z'$$

$$= \frac{c}{2k_g \sin k_g c} \begin{Bmatrix} \sin k_g (c - z) \sin k_g z' \\ \sin k_g z \sin k_g (c - z') \end{Bmatrix} \qquad z \gtrless z' \tag{15}$$

$$\sum_{\ell=0} \frac{1}{K^2 - k^2} \cos k_z z \cos k_z z'$$

$$= \frac{-c}{2k_g \sin k_g c} \begin{Bmatrix} \cos k_g (c - z) \cos k_g z' \\ \cos k_g z \cos k_g (c - z') \end{Bmatrix} \qquad z \gtrless z' \tag{16}$$

where

$$k_g = \left(k^2 - k_c^2\right)^{1/2}$$

An equivalent expression for equation (14) is therefore given by

$$\overline{\overline{G}}_A = \sum_{m,n} C^*_{mn} \left[k_c^2 \bar{\ell}_o \bar{\ell}'_o g_{mn} + \left(\bar{m}_e \bar{m}'_e + \bar{n}_o \bar{n}'_o\right) f_{mn}\right] \tag{17}$$

where

$$C^*_{mn} \qquad \frac{2(2 - \delta_o)}{ab k_c^2 k_g \sin k_g c}$$

$$g_{mn} = \begin{Bmatrix} \cos k_g (c - z) \cos k_g z' \\ \cos k_g z \cos k_g (c - z') \end{Bmatrix} \qquad z \gtrless z' \tag{18}$$

$$f_{mn} = \begin{Bmatrix} \sin k_g (c - z) \sin k_g z' \\ \sin k_g z \sin k_g (c - z') \end{Bmatrix} \qquad z \gtrless z'. \tag{19}$$

Equation (15) is the same as Morse and Feshbach's equation 13.3.47[1] based on the method of scattering superposition starting with dyadic Green's function for $\overline{\overline{G}}_A$ pertaining to a rectangular waveguide.

In principle, once $\overline{\overline{G}}_A$ is known, one can find the vector potential function $\overline{A}$ for any arbitrary current source, including source of the aperture type, as shown by Morse and Feshbach. To find $\overline{E}$, the electric field, another differential operation is needed as

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[1] *P.M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill Book Company, New York (1953).*

[5] *R.E. Collin, Field Theory of Guided Waves, McGraw-Hill Book Company, New York (1960).*

$$\bar{E} = i\omega\left(\bar{A} + \frac{1}{k^2}\nabla\nabla\cdot\bar{A}\right) .$$

If equation (15) is used for $\bar{\bar{G}}_A$, the differential operation involves a series with a discontinuous derivative in the source region that must be executed with due care. For this reason, it appears more appealing to deal with $\bar{\bar{G}}_e$, the dyadic Green's function of the electric type. Once $\bar{\bar{G}}_e$ is known, one can find $\bar{E}$ by applying the formula

$$\begin{aligned}\bar{E}(\bar{R}) &= i\omega\mu_0 \iiint \bar{\bar{G}}_e(\bar{R}|\bar{R}')\cdot\bar{J}(\bar{R}')\, dv' \\ &- \oiint \widetilde{\nabla' x \bar{\bar{G}}_e(\bar{R}'|\bar{R})}\cdot\left[\hat{n} \times \bar{E}(\bar{R}')\right] ds' \end{aligned} \tag{20}$$

where the sign $\sim$ over $\nabla' x \bar{\bar{G}}_e$ denotes the transposition of the entire dyadic function. In fact, it is known[2,3] that

$$\widetilde{\nabla' x \bar{\bar{G}}_{e1}(\bar{R}'|\bar{R})} = \nabla x \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}') \tag{21}$$

where $\bar{\bar{G}}_{e2}$ denotes the dyadic Green's function of the second kind of the electric type, which satisfies the same equation as $\bar{\bar{G}}_{e1}$, the first kind, or simply $\bar{\bar{G}}_e$ in the present designation, but with the boundary condition

$$\hat{n} \times \nabla \times \bar{\bar{G}}_{e2} = 0.$$

A more precisely annotated expression for equation (20), therefore, should be

$$\begin{aligned}\bar{E}(\bar{R}) &= i\omega\mu_0 \iiint \bar{\bar{G}}_{e1}(\bar{R}|\bar{R}')\cdot\bar{J}(\bar{R}')\, dv' \\ &- \oiint \nabla x \bar{\bar{G}}_{e2}(\bar{R}|\bar{R}')\cdot\left[\hat{n} \times \bar{E}(\bar{R}')\right] ds'. \end{aligned} \tag{22}$$

Because of the convenience of using $\bar{\bar{G}}_e$ instead of $\bar{\bar{G}}_A$, it is desirable to give a detailed derivation of the several alternative representations of $\bar{\bar{G}}_e$, which is understood to be $\bar{\bar{G}}_{e1}$ here.

4. EIGEN-FUNCTION EXPANSION OF $\bar{\bar{G}}_e$ FOR A RECTANGULAR CAVITY

By applying the Ohm-Rayleigh method to the equation for $\bar{\bar{G}}_e$ defined by equation (2) or by substituting equation (10) into equation (3), one finds

$$\bar{\bar{G}}_e = \sum_{\ell,m,n} C_{mn} \left[\frac{1}{K^2 - k^2}\left(\bar{M}_{eo}\bar{M}'_{eo} + \bar{N}_{oe}\bar{N}'_{oe}\right) - \frac{k_c^2}{k^2K^2}\bar{L}_{oo}\bar{L}'_{oo}\right] \tag{23}$$

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[2]*C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, PA (1972).*

[3]*C.-T. Tai, Eigen-Function Expansion of Dyadic Green's Functions, Mathematics Note 28, Weapons Systems Laboratory, Kirtland Air Force Base, Albuquerque, NM (July 1973).*

where C_{mn}, $\overline{M}_{eo}$, $\overline{N}_{oe}$, and $\overline{L}_{oo}$ have been defined before. It is observed that the coefficient attached to the $\overline{L}_{oo}\overline{L}'_{oo}$ term is different from the one associated with $\overline{\overline{G}}_A$, while the coefficient attached to the $\overline{M}_{eo}\overline{M}'_{eo}$ and $\overline{N}_{oe}\overline{N}'_{oe}$ terms remains unchanged. They differ because

$$\nabla \cdot \overline{M}_{eo} = 0$$

$$\nabla \cdot \overline{N}_{oe} = 0$$

$$\nabla\nabla\cdot \overline{L}_{oo} = -K^2\overline{L}_{oo}$$

and

$$\frac{k_c^2}{\left(K^2 - k^2\right)K^2}\left(1 - \frac{K^2}{k^2}\right) = -\frac{k_c^2}{k^2K^2}.$$

In terms of the modal functions $\overline{\ell}_o$, $\overline{m}_e$, and $\overline{n}_o$ as defined by equations (11) to (13), one can write equation (23) in the form

$$\overline{\overline{G}}_e = \sum_{\ell,m,n} \frac{C_{mn}}{K^2 - k^2}\Bigg[\overline{m}_e\overline{m}'_e \sin k_z z \sin k_z z'$$

$$+ \frac{k_g^2}{k^2}\overline{n}_o\overline{n}'_o \sin k_z z \sin k_z z'$$

$$+ \frac{k_c^2\left(k^2 - k_z^2\right)}{k^2}\overline{\ell}_o\overline{\ell}'_o \cos k_z z \cos k_z z'$$

$$- \frac{k_z k_c^2}{k^2}\left(\overline{\ell}_o\overline{n}'_o \cos k_z z \sin k_z z' + \overline{n}_o\overline{\ell}'_o \sin k_z z \cos k_z z'\right)\Bigg] \tag{24}$$

where, as before, the constants k_c, k_g, and K are defined by

$$k_c^2 = k_x^2 + k_y^2$$

$$k_g^2 = k^2 - k_c^2$$

$$K^2 = k_x^2 + k_y^2 + k_z^2 = k_c^2 + k_z^2.$$

Now, the series containing the $\overline{\ell}_o\overline{\ell}'_o$ terms has a singular term that can be extracted from the sum. Using equation (19), one finds that

$$\frac{\partial^2 g_{mn}}{\partial z^2} = -k_g^2 g_{mn} - k_g \sin k_g c\,\delta(z - z'). \tag{25}$$

The singular term involving $\delta(z - z')$ results from the discontinuity of $\frac{\partial g_{mn}}{\partial z}$. In view of equations (16) and (18),

$$-\sum_{\ell=1} \frac{k_z^2 \cos k_z z \cos k_z z'}{K^2 - k^2} = \frac{\partial^2}{\partial z^2} \sum_{\ell=0} \frac{\cos k_z z \cos k_z z'}{K^2 - k^2}$$

$$= \frac{c}{2\, k_g \sin k_g c} \frac{\partial^2 g_m}{\partial z^2} = -\frac{c}{2}\delta(z - z') - \frac{k_g c}{2 \sin k_g c} g_{mn}$$

$$= -\frac{c}{2}\delta(z - z') - \sum_{\ell=0} \frac{k_g^2}{K^2 - k^2} \cos k_z z \cos k_z z'.$$

Thus, equation (24) can be written in the form

$$\begin{aligned}
\bar{\bar{G}}_e = &- \sum_{m,n} C_{mn} \left(\frac{k_c}{k}\right)^2 \frac{c}{2}\delta(z-z')\, \bar{\ell}_o \bar{\ell}_o' \\
&+ \sum_{\ell,m,n} \frac{C_{mn}}{K^2 - k^2} \Bigg[\bar{m}_e \bar{m}_e' \sin k_z z \sin k_z z' \\
&+ \frac{k_g^2}{k^2} \bar{n}_o \bar{n}_o' \sin k_z z \sin k_z z' \\
&+ \frac{k_c^4}{k^2} \bar{\ell}_o \bar{\ell}_o' \cos k_z z \cos k_z z' \\
&- \frac{k_z k_c^2}{k^2} \left(\bar{\ell}_o' \bar{n}_o' \cos k_z z \sin k_z z' + \bar{n}_o \bar{\ell}_o' \sin k_z z \cos k_z z' \right) \Bigg]
\end{aligned} \qquad (26)$$

The double series in equation (26) is recognized as $-\frac{1}{k^2} \hat{z}\hat{z}\delta(\bar{R} - \bar{R}')$ because

$$\begin{aligned}
\delta(\bar{R} - \bar{R}') &= \delta(x - x')\delta(y - y')\delta(z - z') \\
&= \sum_{m,n} \frac{4}{ab} \phi_o \phi_o' \delta(z - z')
\end{aligned} \qquad (27)$$

where

$$\phi_o = \sin k_x x \sin k_y y.$$

The triple series can be summed over ℓ by use of equations (15) and (16) and the additional relations

$$\sum_{\ell=1} \frac{k_z \sin k_z \cos k_z z'}{K^2 - k^2} = \frac{-c}{2\, k_g \sin k_g c} \frac{\partial g_{mn}}{\partial z} \qquad (28)$$

$$\sum_{\ell=1} \frac{k_z \cos k_z \sin k_z z'}{K^2 - k^2} = \frac{c}{2 k_g \sin k_g c} \frac{\partial f_{mn}}{\partial z}. \tag{29}$$

The final expression for $\bar{\bar{G}}_e$ after this reduction has the form

$$\begin{aligned}\bar{\bar{G}}_e = &-\frac{\hat{z}\hat{z}\delta(R - R')}{k^2} \\ &+ \sum_{m,n} C^*_{mn} \left[\left(\bar{m}_e \bar{m}'_e + \frac{k_g^2}{k^2} \bar{n}_o \bar{n}'_o \right) f_{mn} \right. \\ &+ \frac{k_c^4}{k^2} \bar{\ell}_o \bar{\ell}'_o g_{mn} + \frac{k_c^2}{k^2} \bar{n}_o \bar{\ell}'_o \frac{\partial g_{mn}}{\partial z} \\ &\left. - \frac{k_c^2}{k^2} \bar{\ell}_o \bar{n}'_o \frac{\partial f_{mn}}{\partial z} \right]\end{aligned} \tag{30}$$

where

$$C^*_{mn} = \frac{2(2 - \delta_o)}{abk_c^2 k_g \sin k_g c}.$$

The functions f_{mn} and g_{mn} are defined by equations (19) and (18).

An alternative procedure to obtain equation (30) is to use the formula

$$\bar{\bar{G}}_e = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla\nabla \right) \cdot \bar{\bar{G}}_A$$

with $\bar{\bar{G}}_A$ given by equation (17). If this procedure is followed, the following relations are needed:

$$\nabla \cdot \left(\bar{m}_e f_{mn} \right) = 0$$

$$\nabla \cdot \left(\bar{n}_o f_{mn} \right) = -k_c^2 \phi_o f_{mn}$$

$$\nabla \cdot \left(\bar{\ell}_o g_{mn} \right) = \phi_o \frac{\partial g_{mn}}{\partial z}$$

$$\nabla\nabla \cdot \left(\bar{n}_o f_{mn} \right) = -k_c^2 \left(\bar{n}_o f_{mn} + \bar{\ell}_o \frac{\partial f_{mn}}{\partial z} \right)$$

$$\nabla\nabla \cdot \left(\bar{\ell}_o g_{mn} \right) = \bar{n}_o \frac{\partial g_{mn}}{\partial z} + \bar{\ell}_o \frac{\partial^2 g_{mn}}{\partial z^2}.$$

From the point of view of waveguide theory, a rectangular cavity can be considered as a waveguide terminated by two conducting walls at the ends. For this reason, it is desirable to identify the significance of equation (30) based on this approach.

The complete expression[3] of the dyadic Green's function of the electric type for an infinite rectangular waveguide is given by

$$\bar{\bar{G}} = -\frac{\hat{z}\hat{z}\delta\left(\bar{R}-\bar{R}'\right)}{k^2} + \sum_{m,n} \frac{i\left(2-\delta_o\right)}{abk_c^2 k_g} \left[\bar{M}\left(\pm k_g\right)\bar{M}'\left(\pm k_g\right) + \bar{N}\left(\pm k_g\right)\bar{N}'\left(\pm k_g\right)\right],\ z \lessgtr z' \tag{31}$$

where

$$\bar{M}\left(k_g\right) = \nabla \times \left[\phi_o e^{ik_g z}\hat{z}\right]$$

$$\bar{N}\left(k_g\right) = \frac{1}{k}\nabla \times \nabla \times \left[\phi_e e^{ik_g z}\hat{z}\right]$$

$$k_c^2 = k_x^2 + k_y^2$$

$$k_g^1 = \sqrt{k^2 - k_c^2}$$

$$\phi_o = \sin k_x x \sin k_y y$$

$$\phi_e = \cos k_x x \cos k_y y.$$

The term $-\dfrac{\hat{z}\hat{z}\delta\left(\bar{R}-\bar{R}'\right)}{k^2}$ was missing in the old treatment,[2] but amended.[3]

For the cavity, $\bar{\bar{G}}_e$ is constructed by the method of scattering superposition. Thus,

$$\bar{\bar{G}}_e = \bar{\bar{G}} + \sum_{m,n} \frac{i\left(2-\delta_o\right)}{abk_c^2 k_g} \left[\bar{M}\left(k_g\right)\bar{A}_1 + \bar{M}\left(-k_g\right)\bar{A}_2 + \bar{N}\left(k_g\right)\bar{B}_1 + \bar{N}\left(-k_g\right)\bar{B}_2\right] \tag{32}$$

where the scattering terms represent the reflected TE and TM modes from the two end walls. After applying the boundary condition $\hat{z} \times \bar{\bar{G}}_e = 0$ at $z = 0$ and $z = c$, one can determine the unknown coefficients $\bar{A}$'s and $\bar{B}$'s. The final result is given by

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[2] *C.-T. Tai, Dyadic Green's Functions in Electromagnetic Theory, Intext Educational Publishers, Scranton, Pa. (1972).*

[3] *C.T.-Tai, Eigen-Function Expansion of Dyadic Green's Functions, Mathematics Note 28, Weapons Systems Laboratory, Kirtland Air Force Base, Albuquerque, N.M. (July 1973).*

$$\bar{A}_1 = \frac{-1}{\sin k_g c} \sin k_g (c-z')\bar{m}'_e$$

$$\bar{A}_2 = \frac{-1}{\sin k_g c} e^{ik_g c} \sin k_g z' \bar{m}'_e$$

$$\bar{B}_1 = \frac{-1}{k \sin k_g c} \left[k_g \sin k_g (c-z')\bar{n}'_o + k_c^2 \cos k_g (c-z') \bar{\ell}'_o \right]$$

$$\bar{B}_2 = \frac{ie^{ik_g c}}{k \sin k_g c} \left[-k_g \sin k_g z' \, \bar{n}'_o + k_c^2 \cos k_g z' \, \bar{\ell}_o \right]$$

where $\bar{m}_e, \bar{n}_o$, and $\bar{\ell}_o$ denote the modal functions defined previously by equations (11) to (13). By summing all the parts in equation (32), it can be shown that the result is identical to equation (30), as it should be. Furthermore, if one introduces the vector wave functions defined by

$$\bar{M}_{eo} \left[k_g z \right] = \nabla x \left[\phi_e \sin k_g z \, \hat{z} \right]$$

$$\bar{M}_{eo} \left[k_g (c-z) \right] = \nabla x \left[\phi_e \sin k_g (c-z) \, \hat{z} \right]$$

$$\bar{N}_{oe} \left[k_g z \right] = \frac{1}{k} \nabla x \nabla x \left[\phi_o \cos k_g z \, \hat{z} \right]$$

$$\bar{N}_{oe} \left[k_g (c-z) \right] = \frac{1}{k} \nabla x \nabla x \left[\phi_o \cos k_g (c-z) \, \hat{z} \right],$$

then equation (30) can be written in the following compact form:

$$\bar{\bar{G}}_e = -\frac{\hat{z}\hat{z}\delta(\bar{R}-\bar{R}')}{k^2}$$

$$+ \sum_{m,n} C^*_{mn} \left\{ \begin{array}{l} \bar{M}_{eo}\left[k_g(c-z)\right] \bar{M}'_{eo}\left[k_g z'\right] - \bar{N}_{oe}\left[k_g(c-z)\right] \bar{N}'_{oe}\left[k_g z'\right] \\ \bar{M}_{eo}\left[k_g z\right] \bar{M}'_{eo}\left[k_g(c-z')\right] - \bar{N}_{oe}\left[k_g z\right] \bar{N}'_{oe}\left[k_g(c-z')\right] \end{array} \right\} z \gtrless z' \qquad (33)$$

where

$$C^*_{mn} = \frac{2(2-\delta_o)}{abk_c^2 k_g \sin k_g c}.$$

The different but equivalent expressions for $\bar{\bar{G}}_e$ given in equations (23), (30), and (33) can be employed in equation (22). Because of the numerical method employed in solving the equation, one might find one form more convenient than the other two. The function $\bar{\bar{G}}_e$ expressed in equation (23) for rectangular, cylindrical, and spherical cavities has been derived by P. Rosenfeld (Ph.D. dissertation), but equations (30) and (31) are not discussed in that work.

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