Mathematics Notes

Note 68

December 1980

Mathematical Foundations of the Singularity and Eigenmode Expansion Method (SEM and EEM)

A.G. Ramm +
University of Michigan
Department of Mathematics
Ann Arbor, MI 48109

Contents:

+ Supported by AFOSR 800204.

Introduction.

This paper is a summary of the invited lecture given by the author at the meeting on mathematical foundations of SEM in November 1980. There were engineers, physicists and mathematicians at this meeting. Thus this paper was written for readers with various interests and backgrounds. The questions under consideration are of practical interest in the fields where wave propagation and scattering are of importance, that is in the fields of unlimited diversity. On the other hand the underlying mathematical theory is deep and relatively new. The mathematical machinery includes the spectral theory of nonselfadjoint operators and pseudo-differential equations on compact manifolds. The mathematical results of use in the EEM and SEM were obtained relatively recently. The author tried to present some of the results and their applications as simply as he could but without making any wrong statements. Whether he succeeded, the reader will tell. The structure of the paper is clear from the contents.

1. Statement of the SEM and EEM.

1.0. SEM and EEM were widely used by physicists and electrical engineers during the last decade [1]-[4],[30]. Their mathematical analysis was started in [5],[6] (see also [7], [8]).

1.1. Let us formulate the EEM. Consider the problem

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad \Omega \,, \tag{1.1}$$

$$u|_\Gamma = f, \tag{1.2}$$

$$r\left(\frac{\partial u}{\partial r} - iku\right) \to 0 \quad \text{as} \quad r \to \infty \,, \tag{1.3}$$

where $k > 0$, $r = |x|$, Ω is an exterior domain with a smooth boundary Γ. Let us look for a solution of (1.1)-(1.3) of the form

$$u = \int_\Gamma G_0(x,s,k)g(s)ds \tag{1.4}$$

where

$$G_0(x,y,k) = \frac{\exp(ik|x-y|)}{4\pi|x-y|} \tag{1.5}$$

and $g(s)$ is an unknown function. The function (1.4) satisfies (1.1) and (1.3). Substituting (1.4) into (1.2) we get the fol-flowing equation for g:

$$Ag = f, \quad Ag = \int_\Gamma G_0(s,s',k)g(s')ds', \quad s \in \Gamma \;. \tag{1.6}$$

Equation (1.6) is an integral equation of the first kind. In Section 3 we will study this equation. At this moment we restrict ourselves by describing the EEM. Suppose that the operator A in (1.6) has eigenvectors:

$$Af_j = \lambda_j f_j \ , \quad |\lambda_1| \geq |\lambda_2| \geq \cdots \tag{1.7}$$

and the set $\{f_j\}$ forms a basis of $H = L^2(\Gamma)$. This means that any element $f \in H$ can be uniquely represented by a convergent in H series

$$f = \sum_{j=1}^{\infty} c_j f_j \ . \tag{1.8}$$

If this assumption is true then one can look for a solution of (1.6) of the form

$$g = \sum_{j=1}^{\infty} g_j f_j \ , \tag{1.9}$$

substitute (1.9) and (1.8) into (1.6) and find the unknown coefficients g_j : $g_j = \lambda_j^{-1} c_j$. Thus

$$g = \sum_{j=1}^{\infty} \lambda_j^{-1} c_j f_j \tag{1.10}$$

is the solution of (1.6). This was the argument used in [1]-[3]. The above method for solving equation (1.6) was called EEM. It was pointed out in [5] that the operator A in (1.6) is non-selfadjoint and therefore it is not obvious that A has eigenvalues. It is even less clear that the eigenvectors of A form a basis of H. Indeed, even in a finite dimensional space a linear operator (a matrix) can have a set of eigenvectors which does not form a basis. For example, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a matrix of an operator on $\mathbb{R}^2$ (two dimensional Euclidean space) then A has only one eigenvector and this vector certainly does not form a basis of $\mathbb{R}^2$. Nevertheless it is known that a root system of a linear operator on $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$. By a root system of a linear operator A we mean the union of the root vectors of A. To construct the root vectors of A we take an eigenvalue λ_j and a corresponding eigenvector f_j and consider the following equations

$$Af_j^{(1)} - \lambda_j f_j^{(1)} = f_j, \dots Af_j^{(r)} - \lambda_j f_j^{(r)} = f_j^{(r-1)} \qquad (1.11)$$

If these equations are solvable but the equation $Af_j^{(r+1)} - \lambda_j f_j^{(r+1)} = f_j^{(r)}$ is not solvable, then the set $(f_j, f_j^{(1)}, \dots, f_j^{(r)})$ is called the Jordan chain associated with the pair (λ_j, f_j), $r+1$ is the length of this chain and $f_j^{(1)}, \dots, f_j^{(r)}$ are called the root vectors of A. If A is a compact operator on a Hilbert space, the definition is the same. It is known [9] that a compact linear operator on a Hilbert space has a discrete spectrum with the only limit point $\lambda = 0$ and the length of any Jordan chain associated with a pair (λ_j, f_j), $\lambda_j \neq 0$ is finite. In a finite dimensional space $\mathbb{R}^n$ the root system of every linear operator forms a basis of $\mathbb{R}^n$. Unfortunately this is not true in the infinite-dimensional Hilbert space. For example the Volterra operator $Vf = \int_0^x f dt$ on $H = L^2[0,1]$ has no eigenvalues. Thus, we face the following basic problems:

1) When does a nonselfadjoint operator A on H have a root system which forms a basis of H?

2) When does the set of eigenvectors of A form a basis of H ?

It is clear that the EEM as described by formula (1.10) is not valid generally speaking because one should take into account the root vectors when writing the series for g and f. This will not make the calculation much more difficult as we will show in Section 3. Therefore from now on we will understand by EEM the solution of (1.6) by means of expansion in series in root vectors of A. Both questions 1), 2) will be discussed in Section 3. It should be mentioned that the specific form of the boundary value problem (1.1)-(1.3) does not play any significant role. We can treat by the same methods the Neumann or the third boundary value problems. What is essential is that the problem in the 3-dimensional unbounded domain Ω is reduced to an equation on 2-dimensional compact manifold (surface Γ).

1.3. We now pass over to the SEM.

Let us consider the problem

$$u_{tt} = \nabla^2 u , \quad t \geq 0, \quad x \in \Omega \tag{1.12}$$

$$u|_\Gamma = 0 \tag{1.13}$$

$$u|_{t=0} = 0 \quad u_t|_{t=0} = f(x) \tag{1.14}$$

If we define

$$v(x,k) = \int_0^\infty \exp(ikt)u(x,t)dt, \tag{1.15}$$

then

$$(\nabla^2 + k^2)v = -f \tag{1.16}$$

$$v|_\Gamma = 0 \tag{1.17}$$

$$r(\frac{\partial v}{\partial r} - ikv) \to 0, \quad r \to \infty \tag{1.18}$$

Thus

$$v(x,k) = \int_{\Omega} G(x,y,k) f(y) dy \tag{1.19}$$

where G is the Green's function for the problem (1.16)-(1.18). We have

$$G = G_0 - \int_{\Gamma} G_0(x,s,k) \frac{\partial G(s,y,k)}{\partial n_s} ds, \tag{1.20}$$

and for $\mu = \frac{\partial G(s,y,k)}{\partial n_s}$ it is easy to get the equation

$$[I + T(k)]\mu = 2 \frac{\partial G_0}{\partial n_s}, \quad T(k)\mu \equiv \int_{\Gamma} \frac{\partial}{\partial n_s} \frac{\exp(ik|s-s'|)}{2\pi|s-s'|} \mu(s') ds' \tag{1.21}$$

From (1.21) it follows that μ is a meromorphic function of k on the whole complex plane k and from this and (1.20) we conclude that $G(x,y,k)$ can be analytically continued as a meromorphic function of k on the whole complex plane k. Moreover the residues of $G(x,y,k)$ (and $\mu(s,y,k)$) are kernels of operators of finite rank (degenerate kernels). This conclusion is an immediate corollary to the following:

<u>Proposition 1.1</u>. Let $T(k)$ be an analytic compact operator function on H for $k \in \Delta$ where Δ is a connected open set in the complex plane. If $I + T(k)$ is invertible at some point $k_0 \in \Delta$ then $(I + T(k))^{-1}$ is finite-meromorphic in Δ.

<u>Remark.</u> Finite-meromorphic means that the Laurent coefficients are operators of finite rank. Though the proposition is well known we will give a short proof in Section 3 for the sake of completeness.

From (1.15) it follows that

$$u(x,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ikt)\, v(x,k)\, dk \tag{1.22}$$

The function $v(x,k)$ is analytic in the half plane $\operatorname{Im} k \geq 0$ and meromorphic in $\operatorname{Im} k < 0$.

Let us introduce the following three conditions:

$$v \text{ is meromorphic (and analytic in } \operatorname{Im} k \geq 0) \tag{1.23}$$

$$|v| \leq c(b)(1 + |k|)^{-a}, \quad a > 1/2, \quad |\operatorname{Re} k| \to \infty, \quad \operatorname{Im} k = b \tag{1.24}$$

where b is an arbitrary constant,

$$|\operatorname{Im} k_j| \to \infty \quad \text{as} \quad j \to \infty \tag{1.25}$$

where $\{k_j\}$ are the poles of v ordered so that $|\operatorname{Im} k_1| \leq |\operatorname{Im} k_2| \leq \ldots$

In (1.23) we can assume that v has a finite number of poles in $\operatorname{Im} k \geq 0$. Also the assumption (1.24) can be relaxed: we can assume that a is an arbitrary fixed number. But we will not discuss these possibilities here. The assumption $a > 1/2$ guarantees that the integral in (1.22) converges in L^2; (1.24)⇒(1.25).

Using (1.23)-(1.25) and moving the contour of integration in (1.22) down we get

$$u(x,t) = \sum_{j=1}^{N} c_j(x,t) e^{-ik_j t} + o(e^{-|\operatorname{Im} k_N| t}), \quad t \to +\infty \tag{1.26}$$

Here

$$c_j(x,t) e^{-ik_j t} = \operatorname*{Res}_{k = k_j} v(x,k) e^{-ikt}, \quad c_j(x,t) = O(t^{m_j - 1}) \tag{1.27}$$

and m_j is the order of the pole k_j. If and only if all the poles are simple, then $c_j(x,t) = c_j(x)$. We have proved

Proposition 1.2. Conditions (1.23)-(1.25) are sufficient for the "asymptotic" SEM.

By asymptotic SEM we mean formula (1.26). BY SEM we will understand the following formula

$$u(x,t) = \sum_{j=1}^{\infty} c_j(x,t) \exp(-ik_j t), \tag{1.28}$$

where the series (1.28) converges uniformly in x and t running through bounded domains. In Section 3 we will show that conditions (1.23)-(1.25) can be verified under relatively general assumptions about the scatterer. Therefore the asymptotic SEM in the form (1.26) can be established. But SEM in the form (1.28) seems to be not established even under very restrictive assumptions about the scatterer. It is an open question:

3) When does (1.28) hold?

2. Discussion of the related questions.

2.1. Interpretation of the EEM as eigenoscillation method with spectral parameter in boundary conditions.

As we pointed out in Section 1 the mathematical idea behind the EEM can be formulated as follows: We substitute the boundary value problem in the exterior 3-dimensional domain by an integral equation over 2-dimensional compact manifold. In Section 3 we will show that this integral equation is a pseudo-differential equation with an elliptic pseudo-differential operator. These terms will be explained in Section 3. Here we want to show that there is a possibility of a physical interpretation of the EEM. Indeed, let the function (1.4) satisfy the following equations

$$(\nabla^2 + k^2)u = 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma \qquad (2.1)$$

$$u^+ = u^-, \quad u = \lambda\left[\left(\frac{\partial u}{\partial n}\right)^+ - \left(\frac{\partial u}{\partial n}\right)^-\right] \quad \text{and (1.3)} \qquad (2.2)$$

where n is the outer unit normal to Γ and $+(-)$ denote the limit value on Γ from inside(outside) of Γ. Since $u|_\Gamma = Ag$, where A is defined in (1.6), and $\left(\frac{\partial u}{\partial n}\right)^+ - \left(\frac{\partial u}{\partial n}\right)^- = g$, the second equality in (2.2) is equivalent to equation $Ag = g$. Thus expansion (1.9) is the expansion in eigenfunction of the problem (2.1)-(2.2) where the spectral parameter λ is in boundary condition. This parameter differs from the usual frequency parameter in the classical approach.

2.2. Complex poles of Green's function (resonances: existence, multiplicity, calculation, stability, asymptotic formulas).

In Section 1 we saw that the complex poles k_j of the Green's function $G(x,y,k)$ are important in SEM (see (1.26)). It seems to be an open question whether the simplicity of the complex poles of G is equivalent to the absence of the root vectors of the operator $T(k)$ defined in (1.21). In [10] it was proved that there are infinitely many purely imaginary poles $i\tau_n$, $\tau_n \to -\infty$, but it is still an open question whether there are infinitely many complex poles of G off the imaginary axis. For one dimensional potential scattering (the Schrodinger equation on the semiaxis) it was proved that the Green's function has infinitely many complex poles (see [11]). This proof can not be carried out for three dimensional potential scattering because it uses essentially the expression of the Green's function in terms of two linearly independent solutions of the Schrodinger equation. It would be interesting to work out a new proof which covers the three dimensional case. It was proved in [12] that the Green's function of the Laplace operator of exterior boundary value problems can be analytically continued on the whole complex plane of k as a meromorphic function. In [13] it was proved that (1.25) holds for the Schrodinger operators with compactly supported potentials. In [14] it was proved for the Laplace operator in the exterior domain and in [15] for the Schrodinger operator in the exterior domain. In [15], estimate (1.24) was introduced and established. Thus we have

Proposition 2.1. The asymptotic SEM in the form (1.26) holds for smooth star-like scatterers.

A body D is called star-like if there exists a point x_0 inside D such that every point on the boundary Γ of D can be seen from x_0.

It is an open question whether the complex poles k_j are simple. Engineers and physicists conjectured that this is the case, but no conclusive arguments were given. For the spherical and linear obstacles the poles are simple but this is due to the fact that the operator A in (1.6) is normal (i.e. $A^*A = AA^*$) if Γ is a sphere or a line [8], [5]. To show that there can be multiple poles of the Green's function of the third boundary value problem consider the following

Example. Let

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad \Omega = \{x: |x| \equiv r \geq 1\}, \ x \in \mathbb{R}^3, \quad k > 0 \qquad (2.3)$$

$$\left(\frac{\partial u}{\partial r} - 2u\right)\Big|_{r=1} = \cos\theta \quad \text{and (1.3) holds} \qquad (2.4)$$

It is easy to find the solution to this problem:

$$u = \frac{-ik \cdot \text{const.}\ r^{-1/2}\, H_{3/2}(kr)\cos\theta}{e^{ik}(k^2 + 4ki - 4)} \qquad (2.5)$$

Thus $k = -2i$ is a pole of order 2. Note that for $k > 0$ the problem (2.3)-(2.4) has a unique solution so that the existence of the multiple pole can not be explained by the presence of active impedance sheet on Γ: the boundary condition (2.4) is passive in the sense that for $k > 0$ the homogeneous problem (2.3)-(2.4) has only trivial solution $u = 0$.

How does one calculate the complex poles? Are they stable under small perturbations of Γ? These questions were answered in [5], [7], [8]. We describe three different approaches given in [8]. The first approach is a general projection method. It was introduced for calculation of the poles in [5]. The complex poles of G are the points at which the operator $I + T(k)$ (see (1.21)) is not invertible (has a nontrivial null space). Let $\{h_j\}$ be a basis of $H = L^2(\Gamma)$, $F_n = \sum_{j=1}^{n} c_j h_j$. We substitute the equation $[I+T(k)]F=0$ by the equation $P_n[I+T(k)]P_nF = 0$, where P_n is the projection on the linear span of $\{h_1,\dots,h_n\}$. This leads to the linear system:

$$\sum_{j=1}^{n} b_{ij}(k)c_j = 0, \quad 1 \le i \le n, \quad b_{ij} = ([I+T(k)]h_i, h_j) \tag{2.6}$$

where $(.,.)$ denotes the inner product in $H = L^2(\Gamma)$. System (2.6) has a nontrivial solution iff

$$\det b_{ij}(k) = 0. \tag{2.7}$$

In the left hand side of (2.7) we have an entire function of k. Let $k_m^{(n)}$ be its zeros.

Proposition 2.2. The set of $k_m = \lim_{n\to\infty} k_m^{(n)}$ coincides with the set of the points at which $I + T(k)$ is not invertible, that is with the union of the set of the complex poles of the Green's function G defined in (1.20) and the spectrum of the interior Neumann problem.

This proposition justifies the current numerical method widely used by engineers for calculation of the complex poles. Its proof given in [5] and has an interesting by-product:

<u>Proposition 2.3.</u> <u>The complex poles depend continuously on the scatterer.</u>

This can be formulated in more detail as follows. Let $x_j = x_j(s_1, s_2)$, $1 \le j \le 3$, $0 \le s_1, s_2 \le 1$, $x_j \in C^2$, $s = (s_1, s_2)$ be a parametric equation of the surface Γ, and $x_j(\varepsilon) = x_j(s) + \varepsilon y_j(s)$, $y_j \in C^2$, $\varepsilon > 0$ is a small parameter, be a parametric equation of the perturbed surface Γ_ε. Let $k_j(k_j(\varepsilon))$ be the complex poles of the Green's function $G(G_\varepsilon)$. Then $k_j(\varepsilon) \to k_j$ as $\varepsilon \to 0$ uniformly for $|k_j| \le R$, where $R > 0$ is arbitrarily large fixed number. For a detailed proof see [8].

The second approach to the calculation of the poles is based on variational principle. Let us consider the set of functions which have the following representation

$$u(x,k) = r^{-1} \exp(ikr) \sum_{j=0}^{\infty} f_j(n,k) r^{-j}, \quad n = x|x|^{-1}, \quad r = |x| \qquad (2.8)$$

$f_0 \not\equiv 0$. The solutions to the Helmholtz equation in an exterior domain satisfying the radiation condition for $k > 0$ satisfy (2.8). It can be proved [16] that if $u(x,k_1)$ and $v(x,k_2)$ belong to the above set, $\mathrm{Re}(k_1 + k_2) \neq 0$, $\pi < \arg k_m < 2\pi$, $m = 1,2$ then the following limit exists

$$< u,v > \equiv \lim_{\varepsilon\to+0} \int \exp(-\varepsilon r \ln r) uv dx, \qquad \int = \int_\Omega , \tag{2.9}$$

and the complex poles of G (defined in 1.19) are the stationary values of the functional

$$k^2 = \text{st} \frac{< \nabla u, \nabla u >}{< u,u >} . \tag{2.10}$$

The admissible functions in (2.10) should satisfy (2.8) and vanish on Γ. Some choice of the basis functions for principle (2.10) is suggested in [16]. The third approach to the calculation of the complex poles is based on the following statement which was proved in [5] (see also [7]).

Proposition 2.3'. The set of the complex poles of G coincide with the set of the complex zeros of the functions $\lambda_n(k)$, where $\lambda_n(k)$ are the eigenvalues of the operator $A(k)$ defined in (1.6) The set of all zeros of the functions $\lambda_n(k)$, $n = 1,2,...$ is the union of the set of complex poles of G and the set of eigenvalues of the Dirichlet Laplacian in the interior domain D with boundary Γ.

For a proof see [8].

Remark [8]. The set of complex poles of G can be also found as the set of the complex roots of the equations $\mu_n(k) = -1$, $n = 1,2,...$ where $\mu_n(k)$ are the eigenvalues of the operator $T(k)$ defined in (1.21).

According to Proposition 2 we can calculate the complex poles by calculating the functions $\lambda_n(k)$ and finding their complex zeros. The eigenvalues $\lambda_n(k)$ can be found by means of the projection method. In Section 3 we given a new variational principle for the spectrum of a compact, nonselfadjoint linear operator on a Hilbert space.

2.3. Mittag-Leffler representation.

From (1.20) and (1.21) if follows that the poles of G coincide with the poles of the operator $(I + T(k))^{-1}$. This operator is a meromorphic function. One can apply the Mittag-Leffler representation to this function. Since in the engineering literature [3] the Mittag-Leffler Theorem was used not quite accurately, we give the statement of the theorem here and discuss the difficulties of its application to our problem (representation of $(I + T(k))^{-1}$).

Proposition 2.4. *Let* $f(k)$ *be a meromorphic function on the whole complex plane* k *and* $|f(k)| \le c\,|k|^p$, $k \in C_n$, *where* C_n *is a proper system of contours and* $p \ge 0$ *is an integer. Let us assume (without loss of generality) that* $k = 0$ *is not a pole of* f. *Then*

$$f(k) = h(k) + \sum_{n=1}^{\infty} [g_n(k) - h_n(k)], \tag{2.11}$$

where

$$h(k) = \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j!} k^j, \quad h_n(k) = \sum_{j=0}^{p} \frac{g_n^{(j)}(0)}{j!} k^j, \tag{2.12}$$

and $g_n(k)$ *is the principal part of* $f(k)$ *at the pole* k_n.

Remark. A proper system of contours $\{C_n\}$ is a system of closed curves such that 1) $k = 0$ lies inside of C_n, 2) $D_n \subset D_{n+1}$, where D_n is the domain inside C_n, 3) $d_n = \text{dist}(0,C_n) \to \infty$ as $n \to \infty$ and $d_n^{-1}|C_n| \leq = c = \text{const}.$ where $|C_n|$ is the length of C_n.

A more general statement is the following Proposition.

Proposition 2.4'. Let $f(k)$ be a meromorphic function. There exists a sequence of integers $p_1,\ldots,p_n,\ldots$ such that (2.11) holds with

$$h_n(k) = \sum_{j=0}^{p_n} \frac{g_n^{(j)}(0)}{j!} k^j . \tag{2.13}$$

Remark. We do not know how fast the numbers p_n in (2.13) grow and therefore it seems impossible to use Proposition 2.4' for numerical calculations. The estimate $\|(I + T(k))^{-1}\| \leq c|k|^p$, $k \in C_n$ is not known, so that (2.11) is also difficult to apply. Even in the case when the poles are simple (so that $g_n(k) = \frac{c_n}{k - k_n}$) we do not know $h_n(k)$ and therefore can not use (2.11). In the engineering literature, sometimes the formula (*) $f(k) = \sum_{n=1}^{\infty} \frac{c_n}{k - k_n}$ was used. This is not correct because the series $\sum_{n=1}^{\infty} g_n$ does not converge in general. Even if $p = 0$, formula (2.11) takes the form

$$f(k) = f(0) + \sum_{n=0}^{\infty} \left(\frac{c_n}{k - k_n} + \frac{c_n}{k_n}\right) \tag{2.14}$$

which differs from (*) .

2.4. Perturbation of complex poles (resonances).

Consider the problem in a general setting. Let $T(k)$ be an analytic compact operator function such that $I + T(k)$ is invertible for some k. Then $(I + T(k))^{-1}$ is finite meromorphic (Proposition 1.1). Let z be a pole of order m of the function $(I + T(k))^{-1}$. Let $T(k,\varepsilon)$ be a compact operator which is analytic on $\{|k - z|\} < a,\ |\varepsilon| < b\}$ and such that $T(k,0) = T(k)$. We want to study the poles of $(I + T(k,\varepsilon))^{-1}$ as functions of ε. Our conclusion is as follows: under a perturbation depending analytically on ε the multiplicity of the pole z cannot increase. It can decrease and the pole $z(\varepsilon)$ of $(I + T(k,\varepsilon))^{-1}$ can have a branch point $\varepsilon = 0$ as a function of ε. It can be represented by Puiseux series, i.e. by a series in the powers of $\varepsilon^{1/r}$ where r is some integer. A proof is given in Section 3.

2.5. Asymptotics of resonances.

In this section we give some asymptotic formulas for the large complex poles nearest to the real axis. Consider the exterior domain Ω and assume that its boundary Γ is smooth and convex and its Gaussian curvature is strictly positive. In the two-dimensional case the following formula for the complex poles of G can be obtained by the method of geometrical optics:

$$K_{pq} \approx \frac{2\pi q}{|\Gamma|}\left(1 - \frac{c\xi_p}{(2\pi q)^{2/3}}\right), \qquad q >> 1, \tag{2.15}$$

where c = const. depends only on the geometry of Γ, $|\Gamma|$ is the length of Γ, $\xi_p = t_p \exp(i\pi/3)$, and $t_p < 0$ are the zeros of the Airy function $v(t) = \pi^{-1/2} \int_0^\infty \cos(ty + \frac{y^3}{3})\,dy$, and p is an integer small in comparison with q.

From (2.15) it follows that

$$|\text{Im}\, k_{pq}| = O(|\text{Re}\, k_{pq}|^{1/3}), \quad q >> 1 \tag{2.16}$$

This estimate can be verified for a circle by direct calculation of the complex poles.

If the boundary Γ is not smooth then instead of (2.16) one can get $|\text{Im}\, k_{pq}| = O(\ln|\text{Re}\, k_{pq}|)$, $q >>1$ (2.17)

Let us explain this by taking a polygon as Γ. The field diffracted by a wedge is proportional to $\exp\left(ikr - \frac{1}{2}\ln(kr)\right)$. Consider a ray having passed once around the polygon. The phase of the field at the point of destination is $ik|\Gamma| - \frac{1}{2}\ln[k^n]$ + terms which do not depend on k. We assume that the polygon has n sides. In order that the field amplitude conserves, one requires that the quantization condition be satisfied:

$$ik|\Gamma| - \frac{n}{2}\ln k = 2\pi q i \tag{2.18}$$

where q is integer. For $q >> 1$, one gets

$$k_q \approx \frac{2\pi q}{|\Gamma|} - \frac{in\ \ln(2\pi q)}{2|\Gamma|}, \quad q >> 1 \tag{2.19}$$

Formula (2.17) follows from (2.19). For the exterior 3-dimensional domain with a smooth convex boundary with positive Gaussian curvature a similar result to (2.15) can be obtained. An additional difficulty in the 3-dimensional problem consists in finding a closed elliptic geodesic $\mathcal{L}$ on Γ. Let s be the length along $\mathcal{L}$, v be the coordinate measured along the geodesic orthogonal to $\mathcal{L}$, $K(s)$ be the Gaussian curvature at the points on $\mathcal{L}$ and $T = |\mathcal{L}|$ is the length of $\mathcal{L}$. Consider the equation

$$(*) \quad \frac{d^2v}{ds^2} + K(s)v = 0, \quad K(s + T) = K(s), \ -\infty < s < \infty.$$

The geodesic $\mathcal{L}$ is called elliptic if equation (*) is stable in the sense of Liapunov. Formulas of this section can be found in [18]. In [18]-[21] some asymptotic formulas for the Green's function as $k \to +\infty$, $\operatorname{Im} k = 0$ are given, but they seem to be of no use in calculating the complex poles. The reason is that the formulas give an expression for the Green's function in terms of exponential functions (geometrical optics) and this expression has no poles.

2.6. Nonsmooth boundaries.

If we want to apply Proposition 1.1 to the problem with a nonsmooth boundary Γ (for example, surface with conical points or edges) we face the following difficulty: the operator $T(k)$ defined in (1.21) is not compact if Γ is not smooth. Potential theory for domains with nonsmooth boundaries was studied in [22]. In this section we will show how to handle the above difficulty. To this end let us first define an essential norm of a linear operator T : $|T|_{ess} = \inf_{Q \in K} \|T - Q\|$, where K is the set of compact operators. Assume that $|T|_{ess} < 1$. Consider the equation $(I + T)g=f$ in a Hilbert space. By our assumption we can write $T = S + Q$ where Q is compact and $\|S\| < 1$. Therefore our equation is equivalent to the equation $(I + Q_1)g = f_1$, $Q_1 = (I + S)^{-1}Q$, $f_1 = (I + S)^{-1}f$, where Q_1 is compact and $(I + S)^{-1}$ is a bijection of H because $\|S\| < 1$ (bijection is a continuous map onto H which has continuous inverse). Therefore equation $(I + T)g = f$ with a noncompact operator T with $|T|_{ess} < 1$ is equivalent to the equation with compact operator. This argument shows that the following generalization of Proposition 1.1 holds.

Proposition 2.5. If $T(k)$ can be represented in the form $T(k) = T + Q(k)$ where $Q(k)$ is analytic and compact, $|T|_{ess} < 1$ and $I + T(k)$ is invertible at some point, then $(I + T(k))^{-1}$ is finite-meromorphic.

In order to apply this proposition, we use the result from [22] which says that $|T(0)|_{ess} < 1$ if the surface Γ is piecewise smooth, has no cusps and its irregular points are conical or the edge of the wedge (in fact in [22] much more general results are given, but they are of no interest to us at this moment. When the surface has cusps we are in trouble, otherwise the theory given in Section 1 holds). We can write $T(k) = T(0) + T(k) - T(0)$. If $Q(k) = T(k) - T(0)$ then $Q(k)$ is analytic and compact and we can use Proposition 2.5. This argument shows that the mermorphic nature of the Green's function holds also in case of non-smooth boundaries without cusps.

2.7. Asymptotics of resonant states. Their orthogonality.

Let $a - ib$, $b > 0$ be a complex pole of G. A resonant state is a solution to the problem:

$$(\nabla^2 + k^2)u = 0; \quad \Omega, k = a - ib, \ b > 0; \ u|_\Gamma = 0, \tag{2.20}$$
satisfying (2.8).

Remark. Radiation condition can not be used for the statement of the problem of finding the resonant state and the corresponding complex poles of the Green's function. Indeed, the problem $(\nabla^2 + k^2)u = 0$ in $\mathbb{R}^3$, where u satisfies (1.3) has a nontrivial solution for any k with $\operatorname{Im} k < 0$. For example, if f is a smooth compactly supported function then

$$u = \int \frac{\exp(ik|x-y|)}{|x-y|} f(y)\,dy - \int \frac{\exp(-ik|x-y|)}{|x-y|} f(y)\,dy$$

is such a solution because the second integral is $O(\exp(-|\operatorname{Im} k|r))$ as $r \to \infty$ and does not change the radiation condition (1.3). The solution of (2.20) satisfies the estimate: $u = O(r^{-1}\exp(br))$. We want to answer the following question: what can be said about u if $u = o(r^{-1}\exp(br))$ as $r \to \infty$? The answer is: $u \equiv 0$ in this case. In order to prove this statement consider the function $v = r\exp(-ikr)u$, $k = a - ib$. By the assumption $v = o(1)$ as

$r \to \infty$. It is easy to see that

$$-v'' - r^{-2}\Delta^* v - 2kv' = 0 \quad \text{for} \quad r > R_0 , \qquad (*)$$

where Δ^* is the angular part of the Laplacian and R_0 is the radius of a sphere containing Γ. Multiplying (*) by v in $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$, integrating in r over (R,∞) and taking the real part, we get

$$0 = \int_R^\infty |v'|^2 \, dr + \int_R^\infty (-\Delta^* v, v) r^{-2} dr + \frac{1}{2} \frac{d|v|^2}{dr}\Big|_{r=R} + b|v|^2\Big|_{r=R} .$$

Thus $\frac{d}{dr}|v|^2 + 2b|v|^2 < 0$, $r > R_0$. This implies that $|v|^2 = O(\exp(-2br))$, $|v| = O(\exp(-br))$ and $u = O(\frac{1}{r})$. Therefore u can be represented by the Green's formula

$$u = \int_\Gamma \overline{G}_0 \frac{\partial u}{\partial N} ds, \quad \overline{G}_0 = \exp(-ikr)/(4\pi r), \quad k = a - ib, \quad b > 0.$$

Therefore $u = O(\exp(-br))$, $u \in L^2(\Omega)$ and $-k^2$ is an eigenvalue of the Dirichlet Laplacian in Ω. Since this operator has no eigenvalues we conclude that $u \equiv 0$.

Let us answer another question: in what sense can the resonant states corresponding to different complex poles k_1 and k_2 be considered as orthogonal?

The answer is : $\langle u(x,k_1), u(x,k_2) \rangle = 0$ where the form $\langle \cdot , \cdot \rangle$ was defined in (2.9). For details see [16] and Section 3.

<u>Remark</u>. In the EEM method we can use the symmetry property of the operator A for finding the coefficients in the expansion

(1.8). Indeed, let us introduce the form $\int fg\,dx \equiv [f,g]$. It is clear that

$$[Af,g] = [f,Ag]. \tag{2.21}$$

If we assume that the eigenvectors f_j of A form a basis of $H = L^2(\Gamma)$, then expansion (1.8) holds and $[f_j,f_m] = 0$ if $j \neq m$. The last statement follows from (2.21): if $Af_j = \lambda_j f_j$, $Af_m = \lambda_m f_m$, $\lambda_j \neq \lambda_m$ then $(\lambda_j - \lambda_m)[f_j,f_m] = [Af_j,f_m] - [f_j,Af_m] = 0$. Thus $[f_j,f_m] = 0$. For $\lambda_j = \lambda_m = \lambda$ one can find linear combinations of the eigenvectors $f_1,\ldots,f_r$ corresponding to λ which are orthogonal with respect to the form $[\cdot,\cdot]$, at least if $[f_j,f_j] \neq 0$, $1 \leq j \leq r$. This can be used for calculation of the coefficients c_j in (1.8) : $c_j = [f,f_j]$.

3. Mathematical Results.

3.1. Justification of EEM. Basisness. Convergence of series in root vectors.

We need some definitons. Let $\{h_j\}$ be an orthonormal basis of H, $m_1 < m_2 < \dots$ a sequence of integers, $m_j \to \infty$ and H_j be the linear span of the vectors $h_{m_j}, h_{m_j+1}, \dots, h_{m_{j+1}-1}$. Let $\{f_j\}$ be a complete minimal system in H and F_j is the linear span of $f_{m_j}, \dots, f_{m_{j+1}-1}$. A system $\{f_j\}$ is called minimal if for any m vector f_m does not belong to the linear span of the remaining vectors $\{f_j\}_{j \neq m}$.

Definition 1. If a linear bijection B exists such that $BH_j = F_j$, $j = 1,2,\dots$ then the system $\{f_j\}$ is called a Riesz basis of H with brackets and we write $\{f_j\} \in R_b(H)$.

Let us remind that B is a bijection if it maps continuously and one-to-one H onto H. By basisness we mean the property of a system of vectors to form a basis of H. A system $\{f_j\} \in R_b(H)$ iff there exist $C_2 \geq C_1 > 0$ such that for any $f \in H$ the inequality (the analogue of the Bessel inequality) holds

$C_1 \|f\|^2 \leq \sum_{j=1}^{\infty} \|P_j f\|^2 \leq C_2 \|f\|^2$, where P_j is the projection onto F_j. We write $A \in R_b(H)\,(A \in R(H))$ if the root system of the linear operator A on H forms a Riesz basis of H with brackets (a Riesz basis of H). Let L be a linear selfadjoint operator on H with discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$ $\lambda_j \to \infty$ as $j \to \infty$. In this case L^{-1} is compact.

Proposition 3.1. Let Q be a (nonselfadjoint) linear operator, $D(Q) \supset D(L)$,

$$\lambda_j = cj^p + O(j^{p_1}) \quad \text{as} \quad j \to \infty,\ p,c > 0,\ p_1 < p , \tag{3.1}$$

$$|L^{-a}Q| \le c_a,\ a < 1;\ p(1-a) \ge 1 . \tag{3.2}$$

Then the spectrum $\sigma(A)$ of the operator $A = L + Q$ is discrete, $\sigma(A) \subset \bigcup_{j=1}^{\infty} \{\lambda: |\lambda-\lambda_j| < |\lambda_j|c_a\}$ where $q > 1$ is an arbitrary number, and $A \in R_b(H)$. If $p(1-a) \ge 2$ and $p_1 < p-1$ then $A \in R(H)$.

A proof of this proposition and some additional information can be found in [8]. Let us show how this proposition can be used in order to prove that $A \in R_b(H)$ and $T(k) \in R_b(H)$, where $A(k)$ and $T(k)$ are defined in (1.6) and (1.21).

We will discuss only $A(k)$ since $T(k)$ can be treated similarly. We have to use some results from the theory of pseudo-differential operators. These results are given in the book [27]. Let us denote by H^q the Sobolev spaces $W^{2,q}(\Gamma)$. If q is a positive integer H^q consists roughly speaking of functions with q derivatives square integrable over Γ . But H^q is defined for any real q . We say that $\text{ord } A = m$ if $A: H^q \to H^{q-m}$, ord A = order of A. By $N(A)$ we denote the null space of A: $N(A) = \{f: Af = 0\}$. We omit some important details and try to explain how to prove that $A(k) \in R_b(H)$.

Let $A = A_0 + A_1$, $A_0 = \text{Re } A$, $A_1 = i\text{Im } A$. The operator $A_0(A_1)$ has the kernel $\frac{\cos(k|x-y|)}{4\pi|x-y|}$ $\left(\frac{i\sin(k|x-y|)}{4\pi|x-y|}\right)$ for $k > 0$.

We assume for simplicity that A^{-1} and A_0^{-1} exist. This assumption is not essential and can be removed at the cost of some additional technical arguments. Let $A_0^{-1} = L$. The operator L is selfadjoint and it can be shown that $\text{ord } L = 1$, $\lambda_j(L) = cj^{1/2} + 0(1)$ as $j \to \infty$. We have $A^{-1} = L + Q$, $Q = -(I+LA_1)^{-1}LA_1L$. The first factor is a bijection and its order is 0 . Thus $\text{ord } Q = 2 \text{ ord } L + \text{ord } A_1 = 2 + \text{ord } A_1$. But A_1 has an infinitely smoothing kernel and therefore $\text{ord } A_1 = -\infty$. Thus $\text{ord } Q = -\infty$. This means that $|L^{-a}Q| < c_a$ for any $a < 1$ (we can take $a < 0$ and $|a|$ as large as we want). From this it follows that conditions (3.1) and (3.2) are satisfied and $A^{-1} \in R_b(H)$. Therefore $A \in R_b(H)$. This argument can be used for complex k also. But in this case the kernels of A_0 and A_1 will be different and in particular, the kernel of A_1 will not be infinitely smoothing. It can be shown that $\text{ord } A_1 = -3$ for complex k .

Now we turn to convergence of the series in root vectors. First we derive some formulas for the coefficients of solution to equation (1.6). Let $g = \sum_{j=1}^{\infty} P_j g$, $f = \sum_{j=1}^{\infty} P_j f$, where P_j is the projection on the root space spanned by the root vectors of A corresponding to the pair $(\lambda_j, f_j^{(0)})$, so that $f_j^{(0)}, \ldots, f_j^{(r_j)}$) is the basis of this root space, $f_j^{(m)}$ are the root vectors. This root space R_j is invariant under the action of A . This means that if $f \in R_j$ then $Af \in R_j$. Therefore $Ag = f$ can be rewritten as $AP_j g = P_j f$, or else

$$\sum_{r=0}^{r_j} g_j^{(r)} A f_j^{(r)} = \sum_{r=0}^{r_j} \ {}_j^{(r)} f_j^{(r)}, \quad j=1,2,\ldots \qquad (*)$$

In what follows, we omit the index j for some time. By the definition of root vectors we have $Af^{(r)} - \lambda f^{(r)} = f^{(r-1)}$, $r \geq 1$, $\lambda = \lambda_j$, $Af^{(0)} = \lambda f^{(0)}$. Therefore from (*) it follows that

$$g_j^{(r_j)} = \frac{c_j^{(r_j)}}{\lambda_j}, \quad g_j^{(m)} = \frac{c_j^{(m)} - g_j^{(m+1)}}{\lambda_j}, \quad m = r_j - 1, \ldots, 0 . \tag{3.3}$$

These recurrent formulas are convenient for the calculation of the coefficients of the expansion of the solution to the equation $Ag = f$ in terms of the root vectors of the operator A. The coefficients $c_j^{(m)}$ corresponding to f are taken to be known. We can rewrite (3.3) as

$$\begin{aligned}
g_j^{(r_j)} &= \frac{c_j^{(r_j)}}{\lambda_j}, \quad g_j^{(r_j-1)} = \frac{c_j^{(r_j-1)}}{\lambda_j} - \frac{c_j^{(r_j)}}{\lambda_j^2}, \\
g_j^{(r_j-2)} &= \frac{c_j^{(r_j-2)}}{\lambda_j} - \frac{c_j^{(r_j-1)}}{\lambda_j^2} + \frac{c_j^{(r_j)}}{\lambda_j^3}, \ldots \\
g_j^{(m)} &= \frac{c_j^{(m)}}{\lambda_j} - \frac{c_j^{(m+1)}}{\lambda_j^2} + \frac{c_j^{(m+2)}}{\lambda_j^3} + \ldots + \frac{(-1)^{r_j-m} c_j^{(r_j)}}{\lambda_j^{r_j-m+1}} .
\end{aligned} \tag{3.4}$$

In order to investigate the rate of convergence of the series in root vectors let us first consider the series in the eigenvectors of the operator L. We note that

$$c_1 \|f\|_{q+1} \leq \|Lf\|_q \leq c_2 \|f\|_{q+1} , \tag{**}$$

where $c_1 \leq c_2$; and $\|\cdot\|_q$ is the norm in H^q. Such type of estimates are well known in the theory of elliptic operators. If $Lf \in H^q$, then its series in eigenvectors of L converges in H^q. Therefore the series for f converges in H^{q+1}.

This argument holds for the series in root vectors provided that (**) holds for A. If A^{-1} exists then $\operatorname{ord} A = 1$, $\operatorname{ord} A^{-1} = -1$, because $A = L(I+L^{-1}Q)$, $A^{-1} = (I+L^{-1}Q)^{-1}L^{-1}$. Therefore (**) holds for A. We conclude that if $f \in H^q$ then its series in the root vectors of A converges in H^q. This means that the smoother f is the better its series in the root vectors converges.

One can estimate the remainder of the series. For example, if $h = Lf \in H^0 = L^2(\Gamma)$, then $\sum_{j=N}^{\infty} c_j f_j = \sum_{j=N}^{\infty} \frac{d_j f_j}{\lambda_j}$ where $Lf_j = \lambda_j f_j$, $f = \sum_{j=1}^{\infty} c_j f_j$, $h = \sum_{j=1}^{\infty} d_j f_j$. Therefore $|\sum_{j=N}^{\infty} c_j f_j| \le \frac{1}{\lambda_N} (\sum_{j=N}^{\infty} |d_j|^2)^{1/2} \le \frac{|h|}{\lambda_N}$. If was proved in [8] that the series in eigenvectors of L and root vectors of A are equiconvergent if $p(1-a) > 2$.

Finally let us note that the root vectors are absent if A is normal, that is $AA^* = A^*A$. This condition can be considered as a condition concerning the surface Γ. It can be written [5] as

$$\int_\Gamma \frac{\sin(k|x-s|-|s-y|)}{|x-s||s-y|}\, ds = 0 \text{ for all } x,y \in \Gamma . \tag{3.5}$$

For the cases when Γ is a sphere or a line the operator A is normal and the EEM method in these cases takes its "engineering" form (without root vectors).

3.2. Justificaton of the Asymptotic SEM.

In Section 1 we gave conditions (1.23)-(1.25) sufficient for the validity of the asymptotic SEM defined in (1.26). Condition

(1.23) was established in Sections 1, 2 under weak restrictions which cover the practical cases. We complete the arguments given in Sections 1, 2 by proving Proposition 1.1. The proof is taken from [12d].

Proof of Proposition 1.1. Let $f_1,\dots,f_n$ be a basis of $N(I+T(z))$, where z is an isolated point where $I + T(z)$ is not invertible. We will show that z is a pole of the operator $(I+T(k))^{-1}$ and its Laurent coefficients are finite rank operators. Consider the operator $B(k) = I + T(k) + \sum_{j=1}^{n}(\cdot,f_j)g_j$, where $\{g_j\}$, $1 \le j \le n$ is a basis of $N(I+T(k)^*)$. Let us show that $N(B(z) = \{0\}$. Indeed, if $B(k)f = 0$, then (*) $(I+T(z)f = \sum_{j=1}^{n}(f,f_j)g_j$. Since $g_j \in N(I+T(k)^*)$ they are orthogonal to $\mathrm{Ran}(I+T(z))$ (Ran A is the range of the operator A). Therefore from (*) we conclude that $(f,f_j) = 0$, $(I+T(z))f = 0$. Since $\{f_j\}$ is a basis of $N(I+T(z))$ we get $f = 0$. Therefore $B^{-1}(z)$ is invertible and $B^{-1}(k)$ is invertible if $|k-z| < \delta$, where $\delta > 0$ is some small number. Equation $(I+T(k))h = f$ is equivalent to $B(k)h = f + \sum_{j=1}^{n}(h,f_j)g_j$, or to the system $h = B^{-1}(k)f + \sum_{j=1}^{n} c_j B^{-1}(k)g_j$, $c_j = (h,f_j)$. From this it follows that

$$\sum_{j=1}^{n} [\delta_{ij}-b_{ij}(k)]c_j = d_i(k),\ 1 \le i \le n, \qquad b_{ij}(k) \equiv (B^{-1}(k)g_j,g_i) \tag{3.7}$$

and $d_i(k) = (B^{-1}(k)f,g_i)$. The functions $b_{ij}(k)$ and $d_i(k)$ are analytic in $|k-z| < \delta$. From Kramer's formulas, it follows that each $a_j(k)$ has a pole at $k = z$ and from (3.6) we can see that the Laurent coefficients of $(I+T(k))^{-1}$ are finite rank operators.

We now turn to conditions (1.24), (1.25). Unfortunately the known proofs of these condiditions given in the papers cited in Section 2.2 are not easy. Therefore we restrict ourselves to a remark concerning Condition (1.24). Suppose that (1.24) holds with some real a (even negative). From the Helmholtz equation (1.16) it follows that (*) $v = -\frac{f}{k^2} - \frac{\nabla^2 v}{k^2}$. Suppose that f is a smooth function which is zero near Γ and near infinity. Then $\nabla^2 v$ satisfies (1.16)-(1.18) with f substituted by $\nabla^2 f$. Therefore $\nabla^2 v$ satisfies inequality (1.24). From this and (*) it follows that v satisfies inequality (1.24) with a substituted by $a + 2$. This argument shows that for smooth and compactly supported in Ω functions f we can estimate (1.24) if we only know that $v(x,k)$ grows not faster than a polynomial as $|\mathrm{Re}\, k| < \infty$, $\mathrm{Im}\, k = \text{const}$.
The idea of all the known proofs of (1.25) is to show that the Green's function is small if $|\mathrm{Re}\, k| \to \infty$ and $|\mathrm{Im}\, k| < \phi(|\mathrm{Re}\, k|)$ where $\phi(r) > 0$ is a nondecreasing function $\phi(r) \to +\infty$ as $r \to \infty$. For the three dimensional potential scattering it was proved in [13] that $\phi(r) = a + b \ln r$, $b > 0$. For diffraction problems in case of a smooth scatterer (Dirichlet or Neumann boundary conditions) $\phi(r) \sim r^{1/3}$ as $r \to \infty$, while for a nonsmooth scatterer $\phi(r) \sim \ln r$ as $r \to \infty$ (see (2.16)(2.17)).

3.3. A variational principle for the spectrum of compact nonselfadjoint operators.

Let T be a compact linear operator on a Hilbert space H with eigenvalues λ_j , $|\lambda_1| \geq |\lambda_2| \geq \dots$. Let $r_j = |\mathrm{Re}\, \lambda_{m(j)}|$ order so that $r_1 \geq r_2 \geq \dots$ and $t_j = |\mathrm{Im}\, \lambda_{n(j)}|$ ordered so that $t_1 \geq t_2 \geq \dots$. The indexes $m(j)$ and $n(j)$ make the ordering. Let L_j be the eigenspace of T corresponding to λ_j , $M_j (N_j)$ be the eigenspace corresponding to $r_j (t_j)$ (that is to $\lambda_{m(j)} (\lambda_{n(j)})$) . Let $\tilde{L}_j = \sum_{m=1}^{j} \dot{+} L_m$, and $\tilde{M}_j, \tilde{N}_j$ are defined similarly. The sign $\dot{+}$ denotes the direct sum. Let $\perp\!\!\perp$ denotes the direct complement in H .

Proposition 3.2. The following formulas hold

$$|\lambda_j| = \max_{x \in \tilde{L}_{j-1}^{\perp\!\!\perp}} \ \min_{\substack{y \in H \\ (x,y)=1}} |(Tx,y)|$$

$$r_j = \max_{x \in \tilde{M}_{j-1}^{\perp\!\!\perp}} \ \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Re}(Tx,y)| ,$$

$$t_j = \max_{x \in \tilde{N}_{j-1}^{\perp\!\!\perp}} \ \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Im}(Tx,y)| .$$

A proof is given in [8]. It would be interesting to try this variational principle numerically.

3.4. Variational principle and perturbation theory for resonances.

In this section we prove existence of the limit (2.9) and orthogonality of the resonant states corresponding to different

k_1, k_2, $\text{Im}\, k_1 < 0$, $\text{Im}\, k_2 < 0$ with respect to the form $< .,. >$ defined in (2.3). Our argument is close to the one in [16]. In order to prove existence of the limit (2.9) it is sufficient to prove existence of the limit

$$< u,v > = \lim_{\varepsilon \to +0} \int_{|x| \geq R} u(x)v(x)\exp(-\varepsilon r \ell n r)dx, \quad r = |x|. \tag{3.8}$$

For $|x| \geq R$ the functions u,v can be represented by the series (2.8). These series converge uniformly in $n \in S^2$ (S^2 is the unit sphere in $\mathbb{R}^3$) and absolutely. Therefore it is sufficient to prove the existence of the limits

$\lim_{\varepsilon \to +0} \int_R^\infty \exp(-\varepsilon\, r \ell n\, r) r^{-j} \cdot \exp(br+iar)dr$, where $b = -\text{Im}(k_1+k_2) > 0$, $a = \text{Re}(k_1+k_2) \neq 0$, $j \geq -2$. Suppose that $a > 0$. Let $C_N = \{z: \ |z-R| = N , \ 0 \leq \arg(z-R) \leq \theta\}$, $C_{\theta N} = \{z: \arg(z-R) = \theta,$ $0 \leq |z-R| \leq N\}$, $C_R = \{z: R \leq z \leq R + N\}$, $C_\theta = C_{\theta\infty}$, $C = C_N \cup C_{\theta N} \cup C_R$. We have $\int_{C_N} \exp(-\varepsilon r \ell n r) r^{-j} \exp(br+iar)dr \to 0$

as $N \to \infty$, $\varepsilon > 0$. Therefore

$$\int_R^\infty \exp(-\varepsilon r \ell n r) r^{-j} \exp(br+iar)dr = \int_{C_\theta} \exp(-\varepsilon r \ell n r) r^{-j} \exp(br+iar)dr \tag{3.9}$$

Let us choose $0 < \theta \leq \frac{\pi}{2}$, such that $\sin\theta > \frac{b}{a}\cos\theta$. Then the integral (3.9) will be absolutely convergent for $\varepsilon = 0$ and (3.8) is proved. The case $a < 0$ is treated similarly with θ substituted by $-\theta$. It is easy to prove the orthogonality of the resonant states, corresponding to $k_1^2 = k_2^2$, with respect to the form (3.8). Indeed, let us multiply the identity $v(\nabla^2+k_1^2)u - u(\nabla^2+k_2^2)v = 0$ in Ω by $\exp(-\varepsilon r \ell n r) \equiv f(r,\varepsilon)$, integrate over $\Omega_R = \{x: \ |x| \leq R,\ x \in \Omega\}$ and take first $R \to +\infty$ and then $\varepsilon \to +0$. Then use the Green's formula. The terms which appear because of the differentiation of $f(r,\varepsilon)$ when we integrate by parts will tend to 0 as $\varepsilon \to 0$. As a result we get $\langle u,v \rangle = 0$. This is what we wanted to show.

Remark. For $a = \mathrm{Re}(k_1+k_2) = 0$ our argument is not valid. We now turn to the proof of the conclusion of Section 2.4. We assume that the operator $I + T(z)$ is not invertible. Let $\phi_1,\ldots,\phi_n$ be an orthonormal basis of $N(I+T(z))$, that is $(\phi_i,\phi_j) = \delta_{ij}$, $\psi_1,\ldots,\psi_n$ be an orthonormal basis of $N(I+T(z)^*)$. Let $Qh = \sum_{j=1}^n (h,\phi_j)\psi_j$. First let us show that the operator $I + T(z) + Q$ is invertible in H. Since $T(z) + Q$ is compact we only need to prove that $N(I+T(z)+Q) = \{0\}$. Suppose that $(I+T(z))h = -\sum_{j=1}^n (h,\phi_j)\psi_j = Qh$. Then by the Fredholm alternative we conclude $(Qh,\psi_i) = 0$, $1 \leq i \leq n$. Thus $(h,\phi_i) = 0$, $1 \leq i \leq n$, $(I+T(z))h = 0$. Therefore $h = 0$. We have proved that $\Gamma = (I+T(z)+Q)^{-1}$ exists. In order to study $(I+T(k,\varepsilon))^{-1}$ let us write

$$(I+T(k,\varepsilon))^{-1} = (I-a(\lambda,\varepsilon))^{-1}\Gamma(\lambda,\varepsilon)$$

where $\Gamma(\lambda,\varepsilon)$ is analytic in $\lambda = k - z$ and ε, $\Gamma(0,0) = \Gamma$ and $a(\lambda,\varepsilon) = (I+T(z)+Q+T(k,\varepsilon)-T(z))^{-1}Q$. Since $a(\lambda,\varepsilon)$ is a finite rank operator (because Q is) we can use a matrix representation of $a(\lambda,\varepsilon)$ and write

$$(I-a(\lambda,\varepsilon))^{-1} = \frac{A(\lambda,\varepsilon)}{\Delta(\lambda,\varepsilon)} .$$

Here $\Delta(\lambda,\varepsilon) = \det(\delta_{ij}-a_{ij}(\lambda,\varepsilon))$, $A = (A_{ji}(\lambda,\varepsilon))$ is the algebraic complement to $\delta_{ji}-a_{ji}(\lambda,\varepsilon)$, $1 \leq i, j \leq n$. By our assumption the operator $\frac{A(\lambda,0)}{\Delta(\lambda,0)}$ has a pole $\lambda = 0$. Let m be its order. This means that $\frac{A(\lambda,0)}{\Delta(\lambda,0)} = \frac{A_0+A_1\lambda+0(\lambda^2)}{\lambda^m(\Delta_0+\lambda\Delta_1+0(\lambda^2))}$, $\Delta_0 \neq 0$, $A_0 \neq 0$. We have

$$A(\lambda,\varepsilon) = A(\lambda,0) + \varepsilon A_1(\lambda,0) + 0(\varepsilon^2) = A_0+\lambda A_1+0(\lambda^2)+\varepsilon A_1(0,0)+\ldots .$$

To the function $\Delta(\lambda,\varepsilon)$ we apply the Weierstrass' preparation theorem [23]. The statement of this theorem is given below for convenience of the reader.

Theorem (Weierstrass' preparation theorem). Let $F(\lambda,\varepsilon)$ be holomorphic in a neighborhood of $(0,0)$, $F(\lambda,0) = \lambda^m f(\lambda)$, $f(0) \neq 0$. Then there exists a holomorphic function $g(\lambda,\varepsilon)$, and holomorphic functions $A_j(\varepsilon)$ such that

$$F(\lambda,\varepsilon) = [\lambda^m + \textstyle\sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j]g(\lambda,\varepsilon),\ A_j(0) = 0,\ 1 \leq j \leq m-1 .$$

From this theorem it follows that $\Delta(\lambda,\varepsilon) = [\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j]g(\lambda,\varepsilon)$, $A_j(0) = 0$. It is now clear that the singularities of $(I-a(\lambda,\varepsilon))^{-1}$ are determined by the function $[\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j]^{-1}$. The equation $\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j = 0$ has $p \leq m$ different roots $\lambda_j(\varepsilon)$, $\lambda_j(0) = 0$. These roots can be represented by the series in powers of $\varepsilon^{1/r}$, where $r > 0$ is some integer. There is an algorithm (method of the Newton diagram) in the literature for construction of these series (Puiseux series) [24]. But our argument has already proved the conclusion of Section 2.4.

4. Examples, comments and some additional material.

4.1. Examples

1. Consider the following matrix $T(k) = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$. This is an analytic operator function on $H = \mathbb{R}^2$. Its resolvent is $(T(k)-\lambda I)^{-1} = \begin{pmatrix} 1-\lambda & -k \\ -k & 1-\lambda \end{pmatrix} / (\lambda^2-1-k^2)$. Its eigenvalues $\lambda_{\pm} = \pm\sqrt{1+k^2}$. For any fixed λ, the resolvent is a meromorphic in k function (as it should be according to Proposition 1.1), but the eigenvalues as functions of k have branch points.

2. A symmetric (with respect to the form $[f,g] = \int f(x)g(x)dx$) nonselfadjoint operator can have root vectors.

Example: $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is an operator on $\mathbb{R}^2$ symmetric with respect to the form $[x,y] = x_1y_1 + x_2y_2$, that is $[Ax,y] = [x,Ay]$. The operator $(A-\lambda I)^{-1}$ has a pole of order 2 at $\lambda = 0$. The corresponding eigenvector is $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and root vector is $\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$.

3. The fact that the algebraic problem to which an original integral equation was reduced (e.g. by a projection method, in particular by the method of moments) has eigenvalues does not guarantee that the original equation has. For example, $Vf = \int_0^x f(t)dt$ has no eigenvalues, but any $n \times n$ matrix has eigenvalues. Proposition 2.2 says that if the original equation has eigenvalues these eigenvalues can be calculated by the projection method described in Section 2. On the other hand if n is a number of the basis functions used in the projection method and $\lambda_j^{(n)}$ is the j-th eigenvalue of the operator $T_n = P_n T P_n$, where P_n is the projection on the n-dimensional

space spanned by the basis functions, then the limit point λ_j as $n \to \infty$ of the sequence $\lambda_j^{(n)}$ is an eigenvalue of T (under weak assumptions about T and the basis functions; e.g. if T is compact and the basis functions form an orthonormal set).

4. There exists an analytic (in k) compact operator $T(k)$ such that $(I+T(k))^{-1}$ has multiple poles but $T(k)$ is diagonalizable for all k that is for any k the operator $T(k)$ has no root vectors. This means that although the EEM (as defined in Section 1) can be applied in the form (1.10), the operator $(I+T(k))^{-1}$ has multiple poles.

Example: $T(k) = \begin{pmatrix} -1+k^2 & 0 \\ 0 & k^2 \end{pmatrix}$, $(I+T(k))^{-1} = \begin{pmatrix} 1/k^2 & 0 \\ 0 & 1/1+k^2 \end{pmatrix}$,

and for any k $T(k)$ is diagonal and therefore has no root vectors.

5. In the finite dimensional space $\mathbb{R}^n$ every linear operator which has n linearly independent eigenvectors is similar to a normal operator; it is diagonal in its eigenbasis (that is in the basis consisting of its eigenvectors). In the Hilbert space there exists an operator with eigenvectors which span H but which is not similar to a normal operator.

An example can be found in [25]. Since this example is rather technical we will not give it here. It seems to be of no practical use for engineers.

6. <u>Whether a root system forms a basis of</u> H <u>or not can depend on the choice of the root system if the total number of the root vectors is infinite</u>.

<u>Example</u>. Let $-y'' = \lambda y$, $0 \le x \le 1$, $y(0) = 0$, $y'(0) = y'(1)$; $H = L^2([0,1])$. The eigenvalues of this problem are $\lambda_n = (2\pi n)^2$, $n = 0,1,2$ and the corresponding eigenvectors and the root vectors are $y_0 = x$, $y_n = \sin 2\pi nx$, $y_n^{(1)} = \frac{x \cos(2\pi nx)}{4\pi n}$. Once can easily check that the biorthogonal system to the above root system is $v_0 = 2$, $v_n = 4(1-x)\sin(2\pi nx)$, $v_n^{(1)} = 16\pi n \cos(2\pi nx)$. Consider now a different choice of the root vectors. Let $z_n^{(1)} = y_n^{(1)} + y_n$. The system biorthogonal to $\{y_n, z_n^{(1)}\}$ is $\{v_n - v_n^{(1)}, v_n^{(1)}\}$. Thus $\|z_n^{(1)}\| \; \|v_n^{(1)} - v_n\| = (1 + 0(\frac{1}{n^2}) \cdot 0(n^2) \to \infty$ as $n \to \infty$. This means that the system $\{y_n z_n^{(1)}\}$ is not a basis of H, because in order for a complete minimal system $\{\phi_n\}$ to be a basis it is necessary that $\sup_n \|\phi_n\| \; \|\psi_n\| \le c$, where $\{\psi_n\}$ is the system biorthogonal to $\{\phi_n\}$. Let us explain the last statement. If $\{\phi_n\}$ is a basis and $\{\psi_n\}$ is the biorthogonal system, that is $(\psi_j, \phi_n) = \delta_{jn}$, then the expansion of an arbitrary element $f \in H$ takes the form $f = \sum_{j=1}^{\infty} c_j \phi_j$ with $c_j = (\psi_j, f)$. The norms of the operators S_n, $S_n f = \sum_{j=1}^{n} (\psi_j, f)\phi_j$ are bounded uniformly in n because $\|S_n f - f\|_{n \to \infty} \to 0$ for any $f \in H$, where $\|\cdot\|$ is the norm of an element of H. Therefore the norm of the operator $S_n - S_{n-1} = (\psi_n, .)\phi_n$ is bounded uniformly in n. But $|(\psi_n, .)\phi_n| = \|\phi_n\| \; \|\psi_n\|$, where $|.|$ is the norm of an operator on H.

We proved that the condition

$$\sup_n \|\phi_n\| \, \|\psi_n\| \leq c \qquad (4.1)$$

is necessary for a complete minimal system to form a basis of H . In (4.1) $\{\psi_n\}$ is the biorthogonal to $\{\phi_n\}$ system. It is known that there exists a unique system biorthogonal to a complete minimal system. The system we gave in example was used in [26].

4.2. Target identification

An interesting problem both theoretically and practically is the inverse problem of identification of the obstacle (target) from the set of complex poles of the Green's function corresponding to this target. No solution to this problem is known. The author thinks that in order to use the complex poles for target identification it is more useful from the practial point of view to have tables of the poles for some typical scatterers (say, aircrafts of various kinds) rather than to use some theoretical results. These few results we will mention below. At present time there is an experimental technique which gives a possibility to find several complex poles corresponding to a given scatterer. It is an interesting theoretical problem to develop an optimization type numerical technique in order to calculate the poles from the experimental data (see Section 5 Problems).
It was observed in [10] that for a star shaped obstacle which contains a ball of radius R_1 and is confined in the ball of radius R_2 the number $N(\tau)$ of the purely imaginary complex

poles $-i\tau_n$, $0 < \tau_n < \tau$ satisfies the inequalities:

$$\frac{1}{2}\left(\frac{R_1}{c}\right)^2 \le \liminf_{\tau \to \infty} \frac{N(\tau)}{\tau^2} , \quad \limsup_{\tau \to \infty} \frac{N(\tau)}{\tau^2} \le \frac{1}{2}\left(\frac{R_2}{c}\right)^2 \tag{4.2}$$

where $c = 0.66274$. Theoretically this gives some information about the scatterer if the asymptotics of large purely imaginary poles is available. But practically one can find from the experimental data only several poles ordered according to the growth of $|\mathrm{Im}\, k_j|$ (poles nearest to the real axis). These poles in general are not purely imaginary. Therefore from the practical point of view it is difficult to make use of (4.2). Let us mention some related results. For interior problem the set of all eigenvalues (which are the poles of the Green's function of the interior problem) does not define the shape of the body uniquely.

For potential scattering on the semi-axis, the set of the poles of the Green's function does not define the potential uniquely. There exists an r-parametric family of potentials having the same set of poles of the Green's functions. Here r is the number of the bound states that is complex poles with positive imaginary parts. Since this observation seems to be new we will give some details. From the theory of the potential scattering for central potentials it is known [11, ch. 12] that the Jost function can be represented in the form

$$f(k) = f(0)\exp(ik\,R) \prod_{n=1}^{\infty} \left(1 - \frac{k}{k_n}\right) , \tag{4.3}$$

where we assume (without loss of generality) that $f(0) \neq 0$. In (4.3) the numbers k_n are the poles of the Green's function of the Schrödinger operator $\ell y = -y'' + V(r)y$, $y(0) = 0$, $0 < r < \infty$. There can be equal poles in (4.3). We assume that $V(r) = 0$ for $r > R$. The Jost function $f(k) = f(0,k)$, where $f(r,k)$ is the solution of the problem $\ell y - k^2 y = 0$, $r > 0$, $y = \exp(ik\, r) + 0(1)$ as $r \to \infty$. Thus if we know the poles of the Green's function we can find $f(k)$ by formula (4.3). If we know $f(k)$ we know the phase shift and the bound states. The phase shift $\delta(k)$ is to be found from the formula

$\exp(2i\delta) = \frac{f(-k)}{f(k)} = S(k)$, where $S(k)$ is the S-matrix [11].

This data and r arbitrary positive parameters (the normalization constants) are sufficient for constructing the potential $V(r)$ which has the above scattering data. The algorithm for the reconstruction of $V(r)$ is well known inverse scattering theory [28]. In particular, the potential $V(r)$ can be uniquely determined from the knowledge of the complex poles iff the imaginary part of each pole is negative.

4.3. Infiniteness of the number of complex poles.

From (4.2), it follows that if the scatterer is star-shaped then its Green's function has infinitely many purely imaginary poles. It is not proved that there are infinitely many complex poles k_j with $\text{Re}\, k_j \neq 0$. Heuristic arguments (e.g. formulas (2.15), (2.18)) show that there are infinitely many such poles. It would be interesting to prove it. For three dimensional scattering for a noncentral potential) this problem is open also.

For potential scattering on the semiaxis it is proved that there are infinitely many complex poles k_j with $\text{Re}\, k_j \neq 0$ [11]. Let us give another proof that there are infinitely many purely imaginary complex poles of the Green's function of the exterior Dirichlet Laplacian. A proof of this statement was given in [10]. Our proof is different, but we use an idea from [10]. Our starting point is Proposition 2.3.

Let $k = -ib$, $b > 0$ be a complete poles. Then the equation $A(b)f \equiv \int_\Gamma G_0(s,s',-ib)f(s')ds' = 0$ has a nontrivial solution, $G_0(s,s',-ib) = \frac{\exp(b|s-s'|)}{4\pi|s-s'|}$. The operator $A(b)$ is selfadjoint in $H = L^2(\Gamma)$ if $b > 0$ and analytic in b. Therefore [29, Ch. 2.§6] its eigenvalues $\lambda_n(b)$ are analytic in b in a neighborhood of the real axis of the complex plane b. If $b \leq 0$ the operator $A(b) > 0$ in H and $\lambda_n(b) > 0$. When $b > 0$ and $b \to +\infty$ the number N_- of the negative eigenvalues of $A(b)$ goes to infinity. Since $\lambda_n(b)$ are analytic in b (continuity would be sufficient for our purpose) they vanish at some point b_n before they become negative. This point b_n is a complex pole according to Proposition 2.3. Therefore if we prove that $N_- \to \infty$ as $b \to \infty$ (*) we prove that there are infinitely many complex poles $-ib_n$. Let us prove (*). Let us take a point inside Γ and draw some lines $\ell_1, \ldots, \ell_n$ intersecting at this point. Let s_n, s_n' be the points of the intersections of ℓ_n with Γ. Let us choose a function h_n which is equal to $1(-1)$ in a small neighborhood $S_n(S'_n)$ of $s_n(s'_n)$ and vanishes outside of these neighborhoods. We assume that $S_n \cap S_m = \emptyset$, $n \neq m$, $S_n \cap S'_n = \emptyset$. In this case the system $\{h_1, \ldots, h_n\}$ is linearly independent. If $(A(b)h,h) < 0$ for $h \in L_n$ and b is sufficiently large then $A(b)$ has at least n negative eigenvalues. Here L_n is the linear span of $\{h_1, \ldots, h_n\}$. If $h = \sum_{j=1}^n c_j h_j$ then $(A(b)h,h) = \sum_{i,j=1}^n a_{ij} c_i c_j$, where

$$a_{ij} = \int_{S_i \cup S'_i} \int_{S_j \cup S'_j} \frac{\exp(b|x-y|)}{4\pi|x-y|} h_i(y) h_j(x)\, dy\, dx =$$

$$= \frac{a^4}{4\pi} \left(\frac{\exp(b|s_i-s_j|)}{|s_i-s_j|} + \frac{\exp(b|s'_i-s'_j|)}{|s'_i-s'_j|} - \frac{\exp(b|s_i-s'_j|)}{|s_i-s'_j|} - \frac{\exp(b|s'_i-s_j|)}{|s'_i-s_j|} \right)$$

where a^2 is the area of S_n, S'_n and

$$i \neq j, \quad a_{ij} \approx -\frac{2\exp(b|s_j-s'_j|)}{4\pi|s_j-s'_j|} a^4 + O\left(\frac{2\exp(ba)}{4\pi a}\right).$$

We can choose lines ℓ_j, $1 \leq j \leq n$ so that $\max_{i \neq j} |s_i - s_j| < \min_j |s_j - s'_j|$. In this case for $b > 0$ sufficiently large the matrix a_{ij} will be negatively definite because the diagonal elements $a_{jj} < 0$, $1 \leq j \leq n$ and dominate if b is sufficiently large. This completes the proof. We make no assumptions about convexity or even star-shapeness of Γ.

<u>Remark</u>. Suppose that Γ_1 and Γ_2 are homothetic and q is the homothety coefficient that is $\Gamma_q = q\Gamma_1$, $q > 1$. Then

$b_j^{(1)} = q\, b_j^{(2)}$ where $-ib_j^{(1)}$ and $-ib_j^{(2)}$, $1 \le j < \infty$ are the purely imaginary poles of the Green's function of the Dirichlet Laplacian in the exterior of Γ_1 and Γ_2 respectively. This can be verified by changing variables $(y_2 = qy_1, x_2 = qx_1)$ in the equation $\int_{\Gamma_1} \frac{\exp(b_j^{(1)}|x_1-y_1|)}{|x_1-y_1|} f(y_1)dy_1 = 0$ corresponding to the pole $-ib_j^{(1)}$.

4.4. Behavior of solutions to wave equation as $t \to +\infty$.

In SEM the information about the behavior of solutions to wave equation as $t \to +\infty$ is obtained (see (1.26)) because some analytic properties of the solution to the corresponding stationary problems are known ((1.23)-(1.25)) . In this section we will point out a general result which says that for a wide class of abstract operators (when analytic continuation of the resolvent kernels of the operators is not necessarily possible) there is a one-to-one correspondence between asymptotic behavior of solution to the abstract wave equation in a Hilbert space

$$u_{tt} + Lu = f \exp(i\omega t) \ , \ u(0) = u_t(0) = 0 \tag{4.4}$$

when $t \to +\infty$ and analytic properties of the resolvent of L in a neighborhood of the spectrum of L . Since these results [17] are of a mathematical nature and their statement is not sufficiently short for including it in this paper, we want only to mention some points of possible interest in applications.

First, all the problems studied in the applications can be formulated as (4.4) with L satisfying conditions from [17]. Second, it is proved in [17] that the limiting amplitude principle is equivalent to the limiting absorption principle. The limiting amplitude principle says that there exists

$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp(-i\omega t) P\, u(t)dt = Pv$, where in applications P is the orthogonal projection on $L^2(\tilde{\Omega})$, where $\tilde{\Omega}$ is a compact subdomain of Ω, and v is the solution to the stationary problem $(L-k^2)v = f$. The limiting absorption principle says that there exists the following limit:

$$\lim_{\varepsilon\to+0} Pv(k+i\varepsilon) = Pv(k),\ v(k+i\varepsilon,f) \equiv [L+(k+i\varepsilon)^2 I]^{-1} .$$

Third, some formula of Tauberian type was proved in [17] but without usual Tauberian conditions (of the type $u(t) \geq 0$ or $u(t) > c$) which are very difficult to verify in practical problems (and theoretical problems in partial differential equations as well). This formula gives a relation between the asymptotic behavior of a function as $t \to +\infty$ and asymptotic behavior of its Laplace transform.

5. Problems.

1) Is is true that $A(k)$, $T(k) \in R(H)$?

In Section 3.1 we proved that $A(k)$, $T(k) \in R_b(H)$. The question is: does basisness without brackets hold?

2) What is the relation between the order of a complex pole and the multiplicity of the zeros of $\lambda_n(k)$? (See proposition 2.3).

3) Can the scatterer be uniquely identified by the set of complex poles of the corresponding Green's function?

4) Prove that there are infinitely many complex poles k_j with $\mathrm{Re}\, k_j \neq 0$ (in diffraction problems and noncentral potential scattering).

5) Are the complex poles of the Green's function of the exterior Dirichlet or Neumann Laplacian simple?

6) Make numerical experiments in the calculation of the complex poles.

7) Prove convergence of the numerical procedure for calculation of the complex poles suggested in [16].

8) Find a theoretical approach optimal in some sense to approximate a function $f(t)$ by the functions of the form $f_N = \sum_{j=1}^{N} \sum_{m=1}^{m_j} \exp(-ik_j t)\, t^{m-1}\, c_{mj}$. Here the numbers c_{mj}, m_j, k_j are to be found so that f_N will approximate $f(t)$ in some optimal way. Currently some methods (e.g. Proney method) are used in practice, but they are not optimal. This problem

seems to be of general interest (optimal harmonic analytis in complex domain)

9) When can SEM in the form of (1.28) be justified?

6. Conclusion.

We hope that it was shown in this paper that:

1) EEM is justfied (in the generalized form of expansion in root vectors).

2) SEM is justified in the asymptotic form (1.26).

3) Numerical projection method for calculation of the complex poles is justified.

4) There are many interesting and difficult open problems in the field.

5) Numerical results and experiments are desirable.

BIBLIOGRAPHY

1. Kacenelenbaum, B.Z.: Expansion of forced oscillations of non-closed systems in eigenfunctions of a discrete spectrum. Radio Eng. and Elect. Phys., 14(1969) 19-23.

2. Baum, C.E.: Emerging technology for transient and broad-band analysis and synthesis of antennas and scatterers. Proc. IEEE 64 (1976) 1598-1616.

3. Selected background on SEM. Research Report, Univ. of Kentucky, 1980.

4. Voitovich, N.; Kacenelenbaum,B.; Sivov, A.: Generalized method of eigenoscillations in diffraction theory (with Appendix written by M.S. Agranovich) Moscow: Nauka 1977 (in Russian).

5. Ramm, A.G.: Exterior problems of diffraction. Radio Eng. Electr. Phys. 17(1972) 1064-1067; MR 51#4864. Computation of quasistationary states in nonrelativistic quantum mechanics. Sov. Phys. Doklady 17(1972) 522-524, MR 56 #14326.

6. Ramm, A.G.: Eigenfunction expansion of a discrete spectrum in diffraction problems. Radio Eng. Electr. Phys. 18(1973) 364-369; MR 50#1641.

7. Ramm, A.G.: Nonselfadjoint operators in diffraction and scattering. Math. Meth. in Appl. Sci. 2(1980) 327-346.

8. Ramm, A.G.: Theory and applications of some new classes of integral equations. New York: Springer, 1980.

9. Dunford, N., Schwartz, J.: Linear operators. New York: Interscience 1958.

10. Lax, P.D.; Phillips, R.S.: Decaying modes for the wave equation in the exterior of an obstacle. Comm. Pure Appl. Math. 22(1969) 737-787.

11. Newton, R.G.: Scattering theory of waves and particles. New York: McGraw Hill, 1966.

12. Ramm, A.G.: a) Analytic continuation of solutions of the Schrödinger equation in spectral parameter and behavior of solutions of nonstationary problem as $t \to +\infty$. Uspehi Mat. Nauk. 19(1964) 192-194.

b): On diffraction problems in domains with infinite boundaries. Proc. 3-rd all-union symposium on wave diffraction. Moscow: Nauka(1964) 28-31.

c): Analytic continuation of resolvent kernel of the Schrödinger operator in spectral parameter and the limiting amplitude principle. Doklady Acad. Nauk, Azerb. SSR 21(1965) 3-7.

d): Some theorems on analytic continuation of the Schrödinger operator resolvent kernel in spectral parameter. Izvestja Acad. Nauk. Armjan. SSR, Mathematics 3(1968) 443-464; MR 42#5563.

13. Ramm, A.G.: Domain free from resonances in the three-dimensional scattering problem. Sov. Math. Doklady, 166(1966) 1319-1322.

14. Lax, P.D.; Phillips, R.S.: A logarithmic bound on the location of the poles of the scattering matrix. Arch. Rat. Mech. Anal. 40(1971) 268-280.

15. Ramm, A.G.: The exponential decay of a solution of a hyperbolic equation. Diff. eq. 6(1970) 1598-1599; MR 44#631.

16. Ramm, A.G.: A variational principle for resonances. Jour. Math. Phys. 21(1980) 2052-2054.

17. Ramm, A.G.: Necessary and sufficient conditions for the validity of the limiting amplitude principle. Izvestija vusov, Mathematics, N5(1978) (Engl. Trans. p. 71-76).

18. Babich, V.; Buldyrev, V.: Asymptotic methods in short wave diffraction problems. Moscow: Nauka 1972 (Russian).

19. Babich, V.; Grigor'eva, N.S.: Nonspectral singularities of Green's function in the exterior of an arbitrary convex polygon. J. of Sov. Math. 11 N5(1979) 676-679.

20. Vainberg, B.: On the short wave asymptotic behavior of solutions of stationary problems. Russ. Math. Survey 30 N2(1975)1-58.

21. Petkov, V.M.: High frequency asymptotics of the scattering amplitude for non-convex bodies. Comm. Part. Diff. Eq. 5(1980)293-329.

22. Burago, Yu, Maz'ya, V.: Potential theory for irregular regions. Seminar in Math. Vol. 3 New York: Consult. Bureau 1969.

23. Golubitsky, M.; Guillemin, V.: Stable mappings and their singularities. New York: Springer 1973.

24. Knopp, K.: Theory of functions. New York: Dover 1945, 1947.

25. Deckard,D; Foias, C.; Pearcy, C.: Compact operators with root vectors that span. Proc. Amer. Math. Soc. 76(1979)101-106.

26. Ilín, V.A.: Existence of the reduced root system of a nonselfadjoint ordinary differential operator. Trudy math. inst. Steklova 142(1976) 148-155 (in Russian)

27. Shubin, M.A.: Pseudo-differential operators and the spectral theory. Moscow: Nauka 1978 (Russian).

28. Chadan, K; Sabatier, P.: Inverse problems in quantum scattering theory. New York: Springer 1977.

29. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-NewYork: Springer 1966.

30. Ramm, A.G.: Theoretical and practical aspects of singularity and eigenmode expansion method. Trans. IEEE A-P., Nov. 1980.