Mathematics Notes

Note 65

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Singular Terms in the Eigen-Function Expansion of Dyadic Green's Function of the Electric Type

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Abstract

The singular terms contained in the eigen-function expansion of the dyadic Green's function of the electric type in the spherical and cylindrical coordinate systems were not properly analyzed in a previous work, [Math Note 28, July, 1973]. The correct form of these terms is summarized in this work.

THE SINGULAR TERMS IN THE EIGEN-FUNCTION EXPANSIONS OF DYADIC GREEN'S FUNCTION OF THE ELECTRIC TYPE

Introduction

The singular terms contained in the eigen-function expansion of the dyadic Green's functions of electric type in the cylindrical and spherical coordinate systems discussed in Reference [1] were not properly assembled. A careful examination of the discontinuous behavior of both the electric and the magnetic dyadic Green's functions shows that the function of the electric type has the general characteristics described by

$$\bar{\bar{G}}_e(\bar{R}|\bar{R}') = \bar{\bar{S}}_e(\bar{R}|\bar{R}') - \frac{k}{k^2}\hat{n}\hat{n}\delta(\bar{R} - \bar{R}') ,$$

where $\hat{n}$ denotes the unit normal to the surface of discontinuity and $\bar{\bar{S}}_e(\bar{R}|\bar{R}')$ the eigen-function expansion of the residue series or integral. We tabulate below the unit normal for various problems.

Problems	Surface of Discontinuity	Unit Normal $\hat{n}$
Waveguides and flat earth	$z = z'$	$\hat{z}$
Circular cylinder and wedge	$r = r'$	$\hat{r}$
Elliptical cylinder	$\xi = \xi'$	$\hat{\xi}$
Sphere and cone	$R = R'$	$\hat{R}$

In this note we show a derivation of these singular terms in the spherical and cylindrical coordinate systems.

THE EIGEN-FUNCTION EXPANSION OF $\bar{\bar{G}}_{eo}$ IN THE SPHERICAL COORDINATE SYSTEM

The expression for $\bar{\bar{G}}_{eo}$ in spherical coordinate systems as given in the Mathematics Note No. 28 [1] was not properly synthesized. The term $\bar{N}_{\substack{e\\o}mn}(K)\bar{N}_{\substack{e\\o}mn}(K')$ in $\nabla \times \bar{\bar{G}}_{mo}$ contains a singular term which was not properly recognized. This error is corrected in the present note.

The expression for $\nabla \times \bar{\bar{G}}_{mo}$ as given by Eq. (2) on p. 35 of the note is

$$\nabla \times \bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^\infty dK \sum_m \sum_n C_{mn} \frac{K^4}{K^2 - k^2}$$

$$\cdot \left[\bar{M}_{\substack{e\\o}mn}(K)\bar{M}'_{\substack{e\\o}mn}(K) + \bar{N}_{\substack{e\\o}mn}(K)\bar{N}'_{\substack{e\\o}mn}(K)\right] , \qquad (1)$$

where

$$C_{mn} = \frac{2 - \delta_o}{2\pi^2} \frac{(2n + 1)}{n(n + 1)} \frac{(n - m)!}{(n + m)!}$$

$$\delta_o = \begin{cases} 1 , & m = 0 \\ 0 , & m \neq 0 \end{cases} .$$

The singular term in (1) can be factored out as follows. We decompose first the function $\bar{N}_{\substack{e\\o}mn}(K)$ into two parts

$$\bar{N}_{\substack{e\\o}mn}(K) = \bar{N}_{\substack{e\\o}mnR}(K) + \bar{N}_{\substack{e\\o}mnt}(K) \qquad (2)$$

similarly for $\bar{N}'_{\substack{e\\o}mn}(K)$. The dyad $\bar{N}_{\substack{e\\o}mnt}(K)\bar{N}'_{\substack{e\\o}mnt}(K)$, which will simply be written as $\bar{N}_t\bar{N}'_t$, is given by

$$\bar{N}_t\bar{N}'_t = \frac{1}{KR}\frac{\partial}{\partial R}[Rj_n(KR)]\frac{1}{KR'}\frac{\partial}{\partial R'}[R'j_n(KR')](\hat{R}\times\bar{m}_{\substack{e\\o}mn})(\hat{R}'\times\bar{m}'_{\substack{e\\o}mn}) \tag{3}$$

where

$$\bar{m}_{\substack{e\\o}mn} = \mp\frac{m}{\sin\theta}P_n^m(\cos\theta)\begin{matrix}\sin\\ \cos\end{matrix}m\phi\hat{\theta} - \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\begin{matrix}\cos\\ \sin\end{matrix}m\phi\hat{\phi}$$

The primed function $\bar{m}'_{\substack{e\\o}mn}$ is defined with respect to the primed variables θ',ϕ'. We now let $KR = \xi$, $KR' = \xi'$, then the product involving the spherical Bessel functions can be written in the form

$$\frac{1}{\xi}\frac{d}{d\xi}[\xi j_n(\xi)]\frac{1}{\xi'}\frac{d}{d\xi'}[\xi' j_n(\xi')]$$

$$= \frac{1}{2n+1}[nj_{n+1}(\xi)j_{n+1}(\xi') + (n+1)j_{n-1}(\xi)j_{n-1}(\xi')]$$

$$-n(n+1)\frac{j_n(\xi)j_n(\xi')}{\xi\xi'} \tag{4}$$

Where we have used the recurrence relations

$$j_n(\xi) = \frac{\xi}{2n+1}[j_{n-1}(\xi) + j_{n+1}(\xi)]$$

and

$$\frac{d}{d\xi}[\xi j_n(\xi)] = \frac{\xi}{2n+1}[(n+1)j_{n-1}(\xi) - nj_{n+1}(\xi)]$$

to convert the product to the form shown in the second line of (4). We now write $\bar{N}_t\bar{N}'_t$ in the form

$$\bar{N}_t\bar{N}'_t = (\bar{N}_t\bar{N}'_t)_1 + (\bar{N}_t\bar{N}'_t)_2 \tag{5}$$

where

$$(\bar{N}_t\bar{N}_t')_1 = \frac{1}{2n+1}[nj_{n+1}(\xi)j_{n+1}(\xi') + (n+1)j_{n-1}(\xi)j_{n-1}(\xi')]$$

$$\cdot (\hat{R} \times \bar{m}_{\substack{e \\ o}mn})(\hat{R}' \times \bar{m}'_{\substack{e \\ o}mn})$$

$$(\bar{N}_t\bar{N}_t')_2 = -n(n+1)\frac{j_n(\xi)j_n(\xi')}{\xi\xi'}(\hat{R} \times \bar{m}_{\substack{e \\ o}mn})(\hat{R}' \times \bar{m}'_{\substack{e \\ o}mn}) \ .$$

Because of the identity

$$\frac{K^4}{K^2 - k^2} = K^2\left[1 + \frac{k^2}{K^2 - k^2}\right]$$

Eq. (1) can now be decomposed into two distinct parts, i.e.,

$$\nabla \times \bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_0^\infty dK \sum_{m,n} C_{mn}K^2[\bar{M}\bar{M}' + (\bar{N}_t\bar{N}_t')_1]$$

$$+ \int_0^\infty dK \sum_{m,n} C_{mn}\left\{\frac{K^2k^2}{K^2 - k^2}\bar{M}\bar{M}' + \frac{K^2k^2}{K^2 - k^2}(\bar{N}_t\bar{N}_t')_1\right.$$

$$\left. + \frac{K^4}{K^2 - k^2}[(\bar{N}_t\bar{N}_t')_2 + \bar{N}_t\bar{N}_R' + \bar{N}_R\bar{N}_t' + \bar{N}_R\bar{N}_R']\right\} \qquad (6)$$

We have omitted the subscript "$\substack{e \\ o}mn$" for clarity. The first integral in (6) represents the singular function $\bar{\bar{I}}_t\delta(\bar{R}-\bar{R}')$, i.e.,

$$\bar{\bar{I}}_t\delta(\bar{R} - \bar{R}') = \int_0^\infty dK \sum_{m,n} C_{mn}K^2[\bar{M}\bar{M}' + (\bar{N}_t\bar{N}_t')_1]$$

because

$$\frac{\bar{\bar{I}}_t\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta} = \frac{\pi}{2}\sum_{m,n} C_{mn}[\bar{m}_{\substack{e\\o}mn}\bar{m}'_{\substack{e\\o}mn} + (\hat{R} \times \bar{m}_{\substack{e\\o}mn})(\bar{R}' \times \bar{m}'_{\substack{e\\o}mn})]$$

$$\int_0^\infty K\, j_n(KR) j_n(KR')\,dK = \frac{\pi}{2}\,\frac{\delta(R - R')}{R^2}$$

and in the spherical coordinate system

$$\delta(\bar{R} - \bar{R}') = \frac{\delta(R - R')\delta(\theta - \theta')\delta(\phi - \phi')}{R^2 \sin\theta} .$$

The second integral in (6) is regular at infinity, hence, it can be evaluated by the method of contour integration, the results gives $k^2\bar{\bar{S}}(\bar{R}|\bar{R}')$ where

$$\bar{\bar{S}}(\bar{R}|\bar{R}') = \frac{ik}{4\pi}\sum_m \sum_n (2 - \delta_o)\frac{2n + 1}{n(n + 1)}\,\frac{(n - m)!}{(n + m)!}$$

$$\cdot \begin{Bmatrix} \bar{M}^{(1)}_{\substack{e\\o}mn}(k)\bar{M}'_{\substack{e\\o}mn}(k) + \bar{N}^{(1)}_{\substack{e\\o}mn}(k)\bar{N}'_{\substack{e\\o}mn}(k) \\ \bar{M}_{\substack{e\\o}mn}(k)\bar{M}'^{(1)}_{\substack{e\\o}mn}(k) + \bar{N}_{\substack{e\\o}mn}(k)\bar{N}'^{(1)}_{\substack{e\\o}mn}(k) \end{Bmatrix} R \gtrless R' \qquad (7)$$

The expression for $\bar{\bar{S}}$ is the same as the one given by Eq. (18), p. 174 of Reference [2]. Thus, we have

$$\nabla \times \bar{\bar{G}}_{mo} = k^2\bar{\bar{S}}_e(\bar{R}|\bar{R}') + \frac{1}{k^2}\bar{\bar{I}}_t\delta(\bar{R} - \bar{R}')$$

hence

$$\bar{\bar{G}}_{eo} = \frac{1}{k^2}[\nabla \times \bar{\bar{G}}_{mo} - \bar{\bar{I}}\delta(\bar{R} - \bar{R}')] = \bar{\bar{S}}_e(\bar{R}|\bar{R}') - \frac{1}{k^2}\hat{R}\hat{R}\delta(\bar{R} - \bar{R}') .$$

Similar analysis applies to the conical case. The dyadic Green's function $\bar{\bar{G}}_{e1}$ should have the form

$$\bar{\bar{G}}_{e1}(\bar{R}|\bar{R}') = \bar{\bar{S}}(\bar{R}|\bar{R}') - \frac{1}{k^2}\hat{R}\hat{R}\delta(\bar{R}-\bar{R}') \tag{8}$$

where $\bar{\bar{S}}(\bar{R}|\bar{R}')$ is defined by Eq. (3), p. 41 of Reference [1] or Eq. (22), p. 192 of Reference [2].

THE EIGEN-FUNCTION EXPANSION IN CYLINDRICAL COORDINATE SYSTEMS

In the cylindrical coordinate system, according to Equations (5.2) of Reference [1], the expression for $\bar{\bar{G}}_{mo}$ using both the continuous spectra of h and λ is

$$\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_{-\infty}^{\infty} dh \int_{0}^{\infty} d\lambda \frac{C_\lambda K}{K^2 - k^2} \left[\bar{M}_{\substack{e\\o}n\lambda}(h)\bar{N}'_{\substack{e\\o}n\lambda}(-h) + \bar{N}_{\substack{e\\o}n\lambda}(h)\bar{M}'_{\substack{e\\o}n\lambda}(-h) \right] \tag{9}$$

where

$$\bar{M}_{\substack{e\\o}n\lambda}(h) = \nabla \times \left[J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi \; e^{ihz}\, \hat{z}\right]$$

$$\bar{N}_{\substack{e\\o}n\lambda}(h) = \frac{1}{K} \nabla \times \bar{M}_{\substack{e\\o}n\lambda}(h)$$

$$K^2 = h^2 + \lambda^2$$

$$C_\lambda = \frac{2 - \delta_o}{4\pi^2\lambda}, \quad \delta_o = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

the primed functions are defined with respect to $\bar{R}'$ or (r',ϕ',z').

Since

$$\bar{\bar{G}}_{eo} = \frac{1}{k^2}[\nabla \times \bar{\bar{G}}_{mo} - \bar{\bar{I}}(\bar{R} - \bar{R}')] \tag{10}$$

in order to find the explicit singularity of $\bar{\bar{G}}_{eo}$ we must search for the singularity term contained in $\nabla \times \bar{\bar{G}}_{mo}$. In Reference [1], it was erroneously stated that $\hat{z}\hat{z}\delta(\bar{R}-\bar{R}')$ was the only singular term contained in $\nabla \times \bar{\bar{G}}_{mo}$. Actually, the correct result should be

$$\nabla \times \bar{\bar{G}}_{mo} = k^2\bar{\bar{S}}_h + (\hat{\phi}\hat{\phi} + \hat{z}\hat{z})\delta(\bar{R} - \bar{R}') \quad . \tag{11}$$

To derive this result it is convenient to examine the discontinuous terms in $\bar{\bar{G}}_{mo}$ and then evaluate these terms involving the differentiation with respect to r. In the first place, the λ-integration in $\bar{\bar{G}}_{mo}$ can be evaluated in a closed form by applying the circuit relation of Bessel functions and the residue theorem, the result yields

$$\bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = \int_{-\infty}^{\infty} dh \sum_n \frac{i(2 - \delta_o)k}{8\pi\eta^2}$$

$$\cdot \left[\bar{N}^{\pm}_{\substack{e\\o}n\eta}(h)\bar{M}^{\pm}_{\substack{e\\o}n\eta}(-h) + \bar{M}^{\pm}_{\substack{e\\o}n\eta}(h)\bar{N}^{\pm}_{\substack{e\\o}n\eta}(-h)\right] \quad , \quad r \gtrless r' \quad , \tag{12}$$

where we have used a new notation in (12), in contrast to the old one used in Reference [1], namely,

$$\bar{M}^{+}_{\substack{e\\o}n\eta}(h) = \nabla \times \left[H_n^{(1)}(\eta r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\, e^{ihz}\,\hat{z}\right] \tag{13}$$

$$\bar{M}^{-}_{\substack{e\\o}n\eta}(h) = \nabla \times \left[J_n(\eta r) \begin{matrix}\cos\\ \sin\end{matrix} n\phi \, e^{ihz} \hat{z} \right] \tag{14}$$

$$\bar{N}^{\pm}_{\substack{e\\o}n\eta}(h) = \frac{1}{k} \nabla \times \bar{M}^{\pm}_{\substack{e\\o}n\eta}(h) \; .$$

$$\eta = \sqrt{k^2 - h^2}$$

The old notation for (13) is $\bar{M}^{(1)}$ and for (14) is just $\bar{M}$. The top line in (12) involving $\bar{N}^{+}\bar{M}'^{-} + \bar{M}^{+}\bar{N}'^{-}$ applies to $r > r'$ and the bottom line involving $\bar{N}^{-}\bar{M}'^{+} + \bar{M}^{-}\bar{N}'^{+}$ applies to $r < r'$. By writing out the explicit expressions for the vector wave functions we found that the following dyads are discontinuous at $r = r'$,

$$\bar{M}^{\pm}_{r}\bar{N}^{\mp}_{r} \; , \quad \bar{M}^{\pm}_{\phi}\bar{N}^{\mp}_{\phi} \; , \quad \bar{M}^{\pm}_{\phi}\bar{N}'^{\mp}_{z} \; , \quad \bar{N}^{\pm}_{r}\bar{N}'^{\mp}_{r} \; , \quad \bar{N}^{\pm}_{\phi}\bar{M}'^{\mp}_{\phi} \; , \quad \bar{N}^{\pm}_{z}\bar{M}'^{\mp}_{\phi} \; .$$

We have omitted the subscript "$\substack{e\\o}n\eta$" for simplicity. We adopt the shorthand notation in regards to the sum of even and odd functions contained in the summation of (1), to

$$\bar{M}^{+}_{\substack{e\\o}n\eta} \bar{N}'^{-}_{\substack{e\\o}n\eta} = \bar{M}^{+}_{en\eta}\bar{N}'^{-}_{en\eta} + \bar{M}^{+}_{on\eta}\bar{N}'^{-}_{on\eta} , \; z > z' \tag{3}$$

and similarly for $\bar{M}^{-}_{\substack{e\\o}n\eta} \bar{N}'^{+}_{\substack{e\\o}n\eta}$. In evaluating $\nabla \times \bar{\bar{G}}_{mo}$ the terms involving the differentiation with respect to r are:

$$\bar{M}_\phi^{\pm}\bar{N}_\phi'^{\mp} \quad , \quad \bar{M}_\phi^{\pm}\bar{N}_z'^{\mp} \quad , \quad \bar{N}_\phi^{\pm}\bar{M}_\phi'^{\mp} \quad \text{and} \quad \bar{N}_z^{\pm}\bar{M}_\phi'^{\mp} \quad .$$

More specifically, they are

$$\left[\frac{1}{r}\ \frac{\partial}{\partial r}\ (rM_\phi^{\pm})\hat{z}\right]\left[N_\phi'^{\mp}\hat{\phi}'\right] \quad , \tag{13}$$

$$\left[\frac{1}{r}\ \frac{\partial}{\partial r}\ (rM_\phi^{\pm})\hat{z}\right]\left[N_z'^{\mp}\hat{z}'\right] \quad , \tag{14}$$

$$\left[\frac{1}{r}\ \frac{\partial}{\partial r}\ (rN_\phi^{\pm})\hat{z}\right]\left[M_\phi'^{\mp}\hat{\phi}'\right] \quad , \tag{15}$$

$$-\left[\frac{\partial N_z^{\pm}}{\partial r}\ \hat{\phi}\right]\left[M_\phi'^{\mp}\hat{\phi}'\right] \quad . \tag{16}$$

The sum of (13) and (15) is continuous at $r = r'$, consequently, they do not produce a singular term. We consider, in detail, the explicit expression for (13) and (15). They are given, respectively, by

$$\frac{1}{r}\ \frac{\partial}{\partial r}\ r\left\{\begin{array}{l} \left(-\dfrac{\partial H_n^{(1)}(\eta r)}{\partial r}\ \begin{matrix}\cos\\ \sin\end{matrix}\ n\phi\right)\left(\pm\ \dfrac{ihnJ_n(\eta r')}{kr'}\ \begin{matrix}\sin\\ \cos\end{matrix}\ n\phi'\right)\ e^{ih(z-z')}\ \hat{z}\hat{\phi}' \\[2ex] \left(-\dfrac{\partial J_n\ (\eta r)}{\partial r}\ \begin{matrix}\cos\\ \sin\end{matrix}\ n\phi\right)\left(\pm\ \dfrac{ihnH_n^{(1)}(\eta r')}{kr'}\ \begin{matrix}\sin\\ \cos\end{matrix}\ n\phi'\right)\ e^{ih(z-z')}\ \hat{z}\hat{\phi}' \end{array}\right\}$$

$$\mp\ \frac{1}{kr}\frac{\partial}{\partial r}\ r\left\{\begin{array}{l} n\ \dfrac{\partial H_n^{(1)}(\eta r)}{\partial r}\ \dfrac{J_n(\eta r')}{r'} \\[2ex] n\ \dfrac{\partial J_n(\eta r)}{\partial r}\ \dfrac{H_n^{(1)}(\eta r')}{r'} \end{array}\right\}\ ih\ \sin\ n(\phi-\phi')e^{ih(z-z')}\ \hat{z}\hat{\phi}'\ ,\ r \gtrless r' \tag{13}$$

and

$$\frac{1}{kr}\frac{\partial}{\partial r} r \left\{ \begin{matrix} \left(\mp \dfrac{ihnH_n^{(1)}(\eta r)}{r} \begin{matrix} \sin \\ \cos \end{matrix} n\phi \right) \left(- \dfrac{\partial J_n(\eta r')}{\partial r'} \begin{matrix} \cos \\ \sin \end{matrix} n\phi' \right) e^{ih(z-z')} \hat{z}\hat{\phi}' \\ \left(\mp \dfrac{ihnJ_n(\eta r)}{r} \begin{matrix} \sin \\ \cos \end{matrix} n\phi \right) \left(- \dfrac{\partial H_n^{(1)}(\eta r')}{\partial r'} \begin{matrix} \cos \\ \sin \end{matrix} n\phi' \right) e^{ih(z-z')} \hat{z}\hat{\phi}' \end{matrix} \right\}$$

$$= \frac{1}{kr}\frac{\partial}{\partial r} r \left\{ \begin{matrix} n \dfrac{H_n^{(1)}(\eta r)}{r} \dfrac{\partial J_n(\eta r')}{\partial r} \\ n \dfrac{J_n(\eta r)}{r} \dfrac{H_n^{(1)}(\eta r')}{\partial r'} \end{matrix} \right\} ih \sin n(\phi-\phi')e^{ih(z-z')} \hat{z}\hat{\phi}' \quad , r \gtrless r' \quad . \tag{15}'$$

Using the recurrence relation for cylindrical functions $Z_n(x)$

$$\frac{\partial Z_n(x)}{\partial x} = \frac{1}{2}[Z_{n-1}(x) - Z_{n+1}(x)]$$

$$\frac{nZ_n(x)}{x} = \frac{1}{2}[Z_{n-1}(x) + Z_{n+1}(x)]$$

one finds

$$n \frac{H_n^{(1)}(\eta r)}{\partial r} \frac{J_n(\eta r')}{r'} = \frac{\eta^2}{4}[H_{n-1}^{(1)}(\eta r) - H_{n+1}^{(1)}(\eta r)][J_{n-1}(\eta r') + J_{n+1}(\eta r')]$$

while

$$n \frac{H_n^{(1)}(\eta r)}{r} \frac{\partial J_n(\eta r')}{\partial r'} = \frac{\eta^2}{4} [H_n^{(1)}(\eta r) + H_{n+1}^{(1)}(\eta r)][J_{n-1}(\eta r') - J_{n+1}(\eta r')] .$$

The sum of these two terms corresponding to the top members within the curly brackets of (13)' and (15)' for $r > r'$, yields

$$\frac{\eta^2}{2} [H_{n+1}^{(1)}(\eta r)J_{n+1}(\eta r') + H_{n-1}^{(1)}(\eta r)J_{n-1}(\eta r')] . \tag{16}$$

Similarly, the sum of the radial functions of the bottom line of (13)' and (15)' yields

$$\frac{\eta^2}{2} [J_{n+1}(\eta r)H_{n+1}^{(1)}(\eta r') + J_{n-1}(\eta r)H_{n-1}^{(1)}(\eta r')] . \tag{17}$$

These two functions are continuous at $r = r'$ although their derivative with respect to r is discontinuous at $r = r'$. In other words, (13) and (15) do not produce any singularity term. We consider now the explicit expression of (14), it is given by

$$\frac{1}{r}\frac{\partial}{\partial r} r \begin{Bmatrix} \left(-\frac{\partial H_n^{(1)}(\eta r)}{\partial r} \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\right)\left(\frac{\eta^2}{k} J_n(\eta r') \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi'\right) e^{ih(z-z')}\hat{z}\hat{z}' \\ \left(-\frac{\partial J_n(\eta r)}{\partial r} \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi\right)\left(\frac{\eta^2}{k} H_n^{(1)}(\eta r') \begin{matrix}\cos\\ \sin\end{matrix}\, n\phi'\right) e^{ih(z-z')}\hat{z}\hat{z}' \end{Bmatrix} r \gtrless r' \; .$$

$$= -\frac{\eta^2}{kr}\frac{\partial}{\partial r} r \begin{Bmatrix} \frac{\partial H_n^{(1)}(\eta r)}{\partial r} J_n(\eta r') \\ \frac{\partial J_n(\eta r)}{\partial r} H_n^{(1)}(\eta r') \end{Bmatrix} \cos n(\phi-\phi')e^{ih(z-z')}\,\hat{z}\hat{z}' \; , \quad r \gtrless z' \; . \tag{19}$$

The function within the curly bracket is now discontinuous at $r = r'$ and its discontinuity is given by the Wronskian of J_n and $H_n^{(1)}$, i.e.,

$$\frac{\partial H_n^{(1)}(\eta r')}{\partial r} J_n(\eta r') - \frac{\partial J_n(\eta r')}{\partial r} H_n^{(1)}(\eta r') = \frac{i2}{\pi r'} \; .$$

When we have a discontinuous function $f^{\pm}(x)$ with a discontinuity $f^{+}(x') - f^{-}(x')$ at $x = x'$, the dèrivative of that function, according to the theory of generalized functions, is interpreted as

$$\frac{df^{\pm}(x)}{dx} = \begin{cases} \frac{df^{+}(x)}{dx} \; , & x > x' \\ [f+(x') - f-(x')]\delta(x - x') \; , & x = x' \\ \frac{df^{-}(x)}{dx} \; , & x > x' \end{cases} \tag{20}$$

The dyad defined by (19) therefore has the explicit value

$$\left\{\begin{array}{l} -\dfrac{\eta^2}{kr}\dfrac{\partial}{\partial r}\left[r\,\dfrac{\partial H_n^{(1)}(\eta r)}{\partial r}\,J_n(\eta r')\right] \\[3ex] -\dfrac{i2\eta^2}{k\pi r}\;\delta(r - r') \\[3ex] -\dfrac{\eta^2}{kr}\dfrac{\partial}{\partial r}\left[r\,\dfrac{\partial J_n(\eta r)}{\partial r}\,H_n^{(1)}(\eta r')\right] \end{array}\right\}\cos n(\phi-\phi')\; e^{ih(z-z')}\;\hat{z}\hat{z}'\;,\quad z \gtrless z' \tag{21}$$

The contribution of the singular term to $\nabla \times \bar{\bar{G}}_{mo}$ corresponds to

$$\int_{-\infty}^{\infty} dh \sum_n \frac{(2 - \delta_o)}{4\pi^2 r}\,\delta(r - r')\cos n(\phi-\phi')\; e^{ih(z-z')}\;\hat{z}\hat{z}'$$

$$= \frac{\delta(r - r')\delta(\phi - \phi')\delta(z - z')}{r}\;\hat{z}\hat{z} \;=\; \hat{z}\hat{z}\delta(\bar{R} - \bar{R}') \tag{22}$$

because we have the representations

$$\int_{-\infty}^{\infty} e^{ih(z-z')}\,dh \;=\; 2\pi\delta(z - z')$$

and

$$\sum_n \frac{2 - \delta_o}{2\pi}\cos n(\phi - \phi') \;=\; \delta(\phi - \phi')\;.$$

Similarly, it can be shown that term (16) contributes a singular term to $\nabla \times \bar{\bar{G}}_{mo}$ in the form $\hat{\phi}\hat{\phi}\delta(\bar{R}-\bar{R}')$. Therefore, we obtain

$$\nabla \times \bar{\bar{G}}_{mo}(\bar{R}|\bar{R}') = k^2 \bar{\bar{S}}_e(\bar{R}|\bar{R}') + (\hat{\phi}\hat{\phi} + \hat{z}\hat{z})\delta(\bar{R}-\bar{R}') , \tag{23}$$

where

$$\bar{\bar{S}}_e(\bar{R}|\bar{R}') = \int_{-\infty}^{\infty} dh \sum_n \frac{2 - \delta_o}{8 \pi\eta^2}$$

$$\cdot \begin{cases} \bar{M}^+_{\substack{e\\o}mn}(h)\bar{M}^-_{\substack{e\\o}mn}(-h) + \bar{N}^+_{\substack{e\\o}mn}(h)\bar{N}'^-_{\substack{e\\o}mn}(-h) , & z > z' \\ \bar{M}^-_{\substack{e\\o}mn}(h)\bar{M}'^+_{\substack{e\\o}mn}(-h) + \bar{N}^-_{\substack{e\\o}mn}(h)\bar{N}'^+_{\substack{e\\o}mn}(-h) , & z < z' \end{cases} \tag{24}$$

In view of (10) the expression for $\bar{\bar{G}}_{eo}$ becomes

$$\bar{\bar{G}}_{eo}(\bar{R}|\bar{R}') = \bar{\bar{S}}_e(\bar{R}|\bar{R}') - \frac{1}{k^2}\hat{r}\hat{r}\delta(\bar{R} - \bar{R}') . \tag{25}$$

This method can also be applied to the spherical case although the extraction of $-\hat{R}\hat{R}\delta(\bar{R}-\bar{R}')/k^2$ directly from the integral representation of $\bar{\bar{G}}_{eo}$ seems to involve less steps when properly done as in this note.

Conclusion

The singularity terms contained in the eigen-function expansion of the dyadic Green's function of the electric type in various coordinate systems have been correctly identified. This author has made repeated errors in the past to extract these terms. It is hoped that with the issue of this amendment to a previous amendment he has redeemed most of the sins.