Mathematics Notes

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Nonselfadjoint Operators in Diffraction and Scattering

A. G. Ramm
University of Michigan

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NONSELFADJOINT OPERATORS IN DIFFRACTION AND SCATTERING

A. G. RAMM

§1. Introduction.

Consider the following problem

$$(\Delta + k^2)u = 0 \text{ in } \Omega, \tag{1}$$

$$\partial u | \partial N = f \text{ on } \Gamma, \tag{2}$$

$$|x|(\partial u/\partial |x| - iku) \to 0 \text{ as } |x| \to \infty, \tag{3}$$

where Ω is an unbounded domain with a smooth closed compact surface Γ, $\Gamma \varepsilon C^2$.

If we look for a solution in the form

$$u = \int_\Gamma \frac{\exp(ikr_{xy})}{4\pi r_{xy}} g(y)dy, \quad r_{xy} = |x-y|, \tag{4}$$

then

$$g = Ag - 2f, \tag{5}$$

where

$$Ag = \int_\Gamma \frac{\partial}{\partial N_t} \frac{\exp(ikr_{ty})}{2\pi r_{ty}} g(y)dy. \tag{6}$$

If the boundary condition is of the form

$$u = f \text{ on } \Gamma, \tag{7}$$

then the integral equation for g takes the form

$$Tg = f, \tag{8}$$

where

$$Tg = \int_\Gamma \frac{\exp(ikr_{ty})}{4\pi r_{ty}} g(y)dy. \tag{9}$$

If one wishes to solve equations (8), (5) by means of expansions in root vectors, one must prove that the root vectors of operators A and T form a basis of $H = L^2(\Gamma)$. Both operators are compact and nonselfadjoint. Apriori it is not clear why these operators have eigenvectors: e.g. Volterra operator has no eigenvectors. In applications it is more convenient to use only eigenvectors, because calculations with the root vectors are more complicated. This leads to the following question: when does a nonselfadjoint operator have no root vectors? Here and below we use the word root vectors meaning associated root vectors. The definition is: if $Ag = \lambda g$, $g \neq 0$, then g is an eigenvector; if equation $Ah_1 - \lambda h_1 = g$ is solvable, then h_1 is an associated vector (or root vector); the set $(g, h_1, \dots h_s)$ is called a Jordan chain with the length s + 1, if $(A - \lambda)\, g = 0$, $(A - \lambda)\, h_1 = g$, $(A - \lambda)\, h_k = h_{k-1}$, $2 \leq k \leq s$, vectors $h_1, \dots h_s$ are called root vectors. An isolated eigenvalue λ is called a normal eigenvalue if its algebraic multiplicity is finite and the Hilbert space H can be decomposed into the direct sum of subspaces $H = L_\lambda \dot{+} R_\lambda$, where L_λ is the root subspace of A and R_λ is an invariant subspace for A in which $(A - \lambda I)^{-1}$ exists. The root subspace L_λ is the linear span of all eigen and root vectors of A corresponding λ. It is well known that λ is a normal eigenvalue iff the projector $P = -(2\pi i)^{-1} \int_{|z-\lambda|=\varepsilon} (A - zI)^{-1} dz$ is finite-dimensional [1]).

If λ is a normal eigenvalue of A then $(A - zI)^{-1} \equiv R_z$ has a simple pole at λ iff the length of the Jordan chain is equal to 1. It means that the eigensubspace of A corresponding to λ coincides with the root subsapce of A corresponding to λ. From the definition given above it follows that the pole λ is simple iff $(A - \lambda I)^2 f = 0 \Rightarrow (A - \lambda I)\, f = 0$.

In physical literature there is a great interest in equations of type (6), (8) and in their counterparts in the electromagentic wave scattering theory [2]. Engineers used the singularity and eigenmode expansion methods for solution of exterior boundary value problems [3], [4]. What they call eigenmode expansion method (EEM) is actually an old Picard's method for solution of selfadjoint integral equations of the first kind. They suppose that the operator T defined by (9) has eigenvectors

$$Tf_j = \lambda_j f_j, \; j = 1, 2, \ldots, |\lambda_1| \geq |\lambda_2| > \ldots, \tag{10}$$

has not root vectors and the set of his eigenvectors $\{f_j\}$ forms a Riesz basis of $H = L^2(\Gamma)$. We remind the reader that $\{f_j\}$ is a Riesz basis of H (or basis equivalent to an orthonormal basis $\{h_j\}$ of H) if a bounded invertible linear operator B exists such that $Bh_j = f_j$. We call an operator B invertible if B^{-1} is bounded and defined on H. Under such an assumption engineers solve equation (8) using the formula

$$g = \sum_{j=1}^{\infty} \lambda_j^{-1}(k)(f,f_j)\, f_j. \tag{11}$$

The following questions are open and of interest to mathematicians:

1) when do the eigenvectors of T and A form a basis of H?

2) when there do not exist root vectors of T and A?

These questions are far from trivial. In fact for the basic equation of the theory of lazers

$$\int_{-1}^{1} \exp\{i(x-y)^2\}\, f(y)dy = \lambda f(x) \tag{12}$$

nothing is known about the existence and properties of its eigen functions until now. Fortunately the situation is much better for the operators

A and T and later we give some reasons for this statement.

The singularity expansion method (SEM) consists in the following. Given the nonstationary problem:

$$\begin{cases} u_{tt} = \Delta u, \ t \geq 0, \ x \epsilon \Omega \subset R^3, \\ \partial u/\partial N = 0 \text{ on } \Gamma, \\ u(0,x) = 0, \ u_t(0,x) = f(x), \end{cases} \tag{13}$$

and assuming

$$v(x,k) = \int_0^\infty \exp(ikt)\ u(x,t)dt \tag{14}$$

we obtain:

$$\begin{cases} \Delta v + k^2 v = -f, \ \partial v/\partial N = 0 \text{ on } \Gamma, \\ (\partial v/\partial |x| - ikv) = o\ (|x|^{-1}) \end{cases} \tag{15}$$

If G(x,y,k) is the Green function for this problem, then

$$v = \int_\Omega G(x,y,k)\ f dy \tag{16}$$

From (14), (16) we obtain

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v \exp(-ikt)dk. \tag{17}$$

For the sake of simplicity we assume that $f \epsilon C_0^\infty(\Omega)$. The function g(x,y,k) can be continued analytically on the whole complex plane k. It is analytic in the upper half plane Imk $\geq$ 0 and meromorphic in the lower half-plane Imk < 0. For details see [5], [6]. Suppose that

$$|v| \leq \frac{C}{1 + |k|^a}, \ a > 0, \ \text{Im}k > -b, \ b > 0, \ a > 0.5. \tag{18}$$

Then we can move down the contour of integration in (17)

$$u(x,t) = \frac{1}{2\pi} \int_{-ic-\infty}^{-ic+\infty} v(x,k) \exp(-ikt)dk, \ 0 < c < b. \tag{19}$$

From this it follows that

$$u(x,t) = \exp(-ct)\, w(x,t), \quad 0 < c < b, \tag{20}$$

where

$$w(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(x,-ic+y) \exp(-iyt)\,dy \tag{21}$$

From (18) it follows that $v \in L^2(\mathbb{R})$ and $w \in L^2(\mathbb{R})$. Suppose that the poles k_j of $v(x,k)$ satisfy the inequality

$$\mathrm{Im}k_j < -F(|\mathrm{Re}k_j|), \tag{22}$$

where $F(x)$ is a continuous positive function,

$$F(x) > 0, \; F(-x) = F(x), \; F(x) \to +\infty \text{ as } x \to \infty. \tag{23}$$

If (18) holds in the domain

$$\mathrm{Im}k_j > -F(|\mathrm{Re}k_j|), \tag{24}$$

then by moving the contour of integration in (17) we get the asymptotic expansion (singularity expansion):

$$u(x,t) = \sum_{j=1}^{n} e^{-ik_j t} v_j(x) + o(e^{-|\mathrm{Im}k_n|t}) \tag{25}$$

This leads to the following questions:

3) What can be said about location of the poles k_j? When does (18) hold? When does (18) hold in the domain (22)?

4) What can be said about the properties of the poles $\{k_j\}$? How to calculate these poles? Do these poles depend continuously on the boundary?

5) To what extent does the set of poles $\{k_j\}$, $\mathrm{Im}k_j < 0$ determine the shape of the obstacle?

These questions are discussed in this paper. They are of interest in applications and difficult from the mathematical point of view.

All of the results concerning operators A and T can be obtained for the analogous integral operators in the electromagnetic wave scattering theory. In what follows formula (1.6) will denote formula (6) in §1. We use autonomous numeration throughout the sections.

§2. When do the eigenvectors of T and A form a basis of H?

1. Bases with brackets. Tests for completeness and basisness.

Let $\{h_j\}$ be an orthonormal basis of H, $m_1 < m_2 \ldots$ a sequence of integers, $m_\ell \to \infty$ as $\ell \to \infty$, and let H_ℓ be the linear span of the vectors $h_{m_\ell}, h_{m_\ell+1}, \ldots h_{m_{\ell+1}-1}$. Let $\{f_j\}$ be a complete minimal system in H, and F_ℓ be the linear span of vectors $f_{m_\ell}, \ldots f_{m_{\ell+1}-1}$. By basisness we mean the property of a system of vectors or subspaces to form a basis of H.

Definition 1. If a linear, bounded, invertible operator B exists such that $BH_\ell = F_\ell$ then the system $\{f_j\}$ is called a Riesz basis of H with brackets (notation: $\{f_j\} \varepsilon R_b(H)$).

Remark 1. It is known [1], that $\{f_j\} \varepsilon R_b(H)$ iff $C_1 |f|^2 \leq \sum_{\ell=0}^{\infty} |P_\ell f|^2 \leq C_2 |f|^2$, where $|\cdot|$ is the norm in H, $C_2 \geq C_1 > 0$ are constants, P_ℓ is the projector on F_ℓ, $f \varepsilon H$ is an arbitrary element of H. Projector P_ℓ is defined by the direct decomposition $H = F_\ell \dot{+} G_\ell$, where G_ℓ is the union of the subspaces F_j for $j \neq \ell$.

Definition 2. Denote by Q_ℓ the orthoprojector on H_ℓ. If $\sum_{\ell=0}^{\infty} |P_\ell - Q_\ell|^2 < \infty$ then the system $\{f_j\}$ is called a Bari basis with brackets (notation $\{f_j\} \varepsilon B_b(H)$).

Definition 3. A linear closed densely defined operator L on a Hilbert space H is called an operator with discrete spectrum iff its spectrum $\sigma(L)$ consists only of normal eigenvalues λ_j, $|\lambda_1| \le |\lambda_2| \le \dots |\lambda_j| \le \dots$, $|\lambda_j| \to \infty$ as $j \to \infty$.

Remark 2. If L is a normal operator with discrete spectrum, $0 \notin \sigma(L)$, then L^{-1} is compact.

In what follows we assume for the sake of simplicity that L is a selfadjoint operator with a discrete spectrum $\{\lambda_j\}$, $0 \notin \sigma(L)$,

$$\lambda_j = cj^p + O(j^{p_1}), \text{ as } j \to \infty, \ p > 0, \ c > 0, \ p_1 < p. \tag{1}$$

Consider the operator

$$A = L + Q \tag{2}$$

where Q is a (nonselfadjoint) linear operator,

$$|L^{-a}Qf| \le C_a|f|, \ \forall f \in H, \ a < 1, \ D(Q) \supset D(L). \tag{3}$$

Since

$$(L + Q - \lambda I)^{-1} = \{I + (L - \lambda I)^{-1}L^a L^{-a}Q\}^{-1} (L - \lambda I)^{-1} \text{ for } \lambda \notin \sigma(L) \tag{4}$$

it is clear, that

$$\lambda \notin \sigma(A) \text{ if } |(L - \lambda I)^{-1}L^a| < C_a^{-1}. \tag{5}$$

It is clear that

$$|(L - \lambda I)^{-1}L^a| \le \sup_j |\lambda_j - \lambda|^{-1}|\lambda_j|^a. \tag{6}$$

If $|\lambda_j - \lambda| \ge |\lambda_j|^a C_a q$, where $q > 1$ is arbitrary, then (5) holds. Hence we have proved the main part of the following lemma.

Lemma 1. Suppose that L is a selfadjoint operator with a discrete spectrum, Q is a linear operator, $A = L + Q$, and (3) holds. Then $\sigma(A) \subset K$, where

$$K = \bigcup_{j=1}^{\infty} \{\lambda : |\lambda - \lambda_j| < |\lambda_j|^a C_a q, \ q > 1\}, \tag{7}$$

and $\sigma(A)$ is discrete.

Proof. It remains to prove the last statement of Lemma 1. The statement follows immediately from the compactness of $(L - \lambda I)^{-1}$ and boundedness of the operator $\{I + (L - \lambda I)^{-1}L^{a}L^{-a}Q\}^{-1}$ in (4).

Remark 3. Estimates of type (6) were used earlier by Kacnelson [18], [3]. We made no use of assumption (1) so far.

The following theorem is due to Kacnelson [18], [3].

Theorem 1. Under the assumptions (1), (3) $A \in R_b(H)$ if $p(1-a) = 1$, and $A \in B_b(H)$ if $p(1 - a) > 1$.

Remark 4. We write $A \in R_b(H)$ $(B_b(H))$ if the root vectors of A form a Riesz (Bari) basis of H with brackets.

Remark 5. Actually for Theorem 1 to be true it is sufficient to use the following estimate instead of (1): $\lambda_j \geq cj^p$ (see [18]).

Remark 6. Under some additional assumptions M. S. Agranovich proved that the series in root vectors of A converges rapidly (see Appendix in [3]).

Remark 7. Completeness of the root system of a linear operator A in a Hilbert space H can be proved by means of the following theorems.

Theorem 2 ([1]) If L is a selfadjoint operator on a Hilbert space H with a discrete spectrum, $0 \notin \sigma(L)$, Q is a linear operator $D(Q) \supset D(L)$, $L^{-1}Q$ is compact and $p(L^{-1}QL^{-1}) < \infty$, then the system of root vectors of $A = L + Q$ is complete in H.

Remark 8. The symbol $p(A) < \infty$ means that A is compact and $\sum_1^\infty s_n^p < \infty$, where $s_n = \lambda_n \{(A^*A)^{1/2}\}$ are the s-values of A.

Theorem 3([1]). The system of root vectors of a compact dissipative operator A with nuclear imaginary component is complete in H if $\liminf ns_n(A) = 0$.

Remark 9. A linear operator A is called dissipative if $\text{Im}\,(Af, f) \geq 0$ $\forall f \in D(A)$. A compact linear operator is called nuclear if $\sum_1^\infty s_n(A) < \infty$.

Theorem 4[11]. If $A \geq 0$ is compact, B is dissipative and nuclear then the root system of A + B is complete in H.

Example 1[11]. Operator (1.9) can be split into the sum $T = T_0 + T_1$, where $T_0 g = \int_\Gamma (4\pi r_{ty})^{-1} g(y)dy$, $T_0 > 0$, and $T_1 = T - T_0$ is nuclear and dissipative. The last statements is easy to verify (see [11] for details). Thus from Theorem 4 it follows that the root system of operator (1.9) is complete in $H = L^2(\Gamma)$. Actually this system forms a Riesz basis as we shall prove later.

2. Elliptic pseudo-differential operators (PDO) on Γ.

In order to explain how to prove that the root systems of operators A (formula (1.6)) and T (formula (1.9)) form a Riesz basis of H we start with the operator T. It is clear that

$$T = T_0 + T_1$$

where T_0, T_1 are defined in Example 1. It is easy to verify that T_0 is an elliptic pseudo-differential operator on Γ of order −1 and T_1 is a PDO of order $\gamma < -1$ (in fact $\gamma = -3$). Suppose that $\text{Ker}\, T_0 = \{0\}$. Then $L = T_0^{-1}$ exists, L is a selfadjoint operator with discrete spectrum. If $\text{Ker}\, T = \{0\}$, then

$$(T_0+T_1)^{-1} = (I+LT_1)^{-1}L = L + Q, \text{ where } Q = -(I+LT_1)^{-1}LT_1L, \ |L^{-a}Q| \leq C \text{ for } a = 2 + \gamma < 1 \tag{8}$$

because $\text{ord}\, LT_1L = 2 + \gamma < 1$. Condition (1) is valid for PDO under very general assumptions [20]. Therefore one can apply Theorem 1 and obtain

Proposition 1. The root system of operator T defined by formula (1.9) forms a Riesz basis in $H = L^2(\Gamma)$.

Remark 10. It is easy to verify that Ker $T_0 = \{0\}$ and Ker $T = \{0\}$.

Remark 11. One can find e.g. in [21] how to calculate the order of an elliptic PDO.

Remark 12. It is possible (and in a way more reasonable) to choose $T_0 = 0.5\,(T + T^*)$, because in this case T_1 will be of the order $-\infty$ for real $k > 0$ since the kernel of T_1 is $\frac{\sin kr_{ty}}{r_{ty}} \varepsilon C^\infty$.

Consider now the operator A defined by formula (1.6).

It is easy to verify that A is a pseudo-differential elliptic operator, and ord $A = -1$. If $A_0 = 0.5\,(A + A^*)$, $A_1 = A - A_0$, then ord $A_0 = -1$, ord $A_1 < -1$. If ker $A_0 = \{0\}$, and ker $A = \{0\}$ one can use the arguments similar to ones used above and obtain the analogue of Proposition 1 for the operator A. If ker $A_0 \neq \{0\}$ then dim ker $A_0 < \infty$ and ker $A_0 \subset C^\infty$. This statement follows from the apriori estimates for elliptic PDO [21]. Thus, one can add a finite dimensional operator P to A_0 and subtract this operator from A_1. Since ker $A_0 \subset C^\infty$ operator P can be chosen so that ord $(A_0 + P) =$ ord $A_0 = -1$ (ord $P = -\infty$), and ker $(A_0 + P) = \{0\}$. Hence, one can assume that ker $A_0 = \{0\}$. If ker $A_0 = \{0\}$ then A_0^{-1} exists and has a discrete spectrum. Since ord $A_1 <$ ord A_0 the operator $A_0^{-1} A_1$ is compact in H. From this argument and the formula $A = A_0\,(I + A_0^{-1} A_1)$ it follows that the root subspace N of A corresponding to $\lambda = 0$ is finite-dimensional. Therefore one can split H into a direct sum $H = N \dot{+} M$, where N and M are invariant subspaces of A and ker $A|_M = \{0\}$, $A|_M$ denoting the restriction of A to M. Hence, one can assume that ker $A = \{0\}$. This completes the proof of the following proposition.

Proposition 2. The root system of operator A defined by formula (1.6) forms a Riesz basis of H.

§3. When do T and A have no root vectors?

1. A simple sufficient condition was given in [11]: in order that T (or A) has no root vectors it is sufficient that T is normal. This condition $T^*T = TT^*$ can be written explicitly [11] and it is a condition concerning the surface Γ. In [11] it was verified that for operator T this condition is satisfied if Γ is sphere. For linear antenna this condition is also satisfied [11]. Of course, this condition is not necessary. In a finite-dimensional Hilbert space H every linear operator A without root vectors is similar to a normal operator. Indeed, if A has no root vectors then its eigenvectors $\{f_j\}$ form a basis of H. If $\{h_j\}$ is an orthonormal basis of H, $Af_j = \lambda_j f_j$ and $f_j = Ch_j$, then $C^{-1}ACh_j = \lambda_j h_j$. It means that operator $C^{-1}AC$ is normal.

In infinite-dimensional Hilbert space H this is not true: there exist compact operators whose eigenvectors span H but these operators are not similar to a normal operator (an example is given in [24]).

2. In [23] the following observation was formulated: the eigen-subspace and the root subspace of a compact operator T, corresponding to the number λ, coincide iff 1) λ is a simple pole of the resolvent $(T - \lambda I)^{-1}$, or iff 2) $(T - \lambda I)^2 f = 0 \quad (T - \lambda I)f = 0$, or iff 3) the operator $T - \lambda I$ does not have zeros in the subspace $R(T - \lambda I)$, where $R(A)$ denotes the range of A.

§4 What can be said about the location and properties of the complex poles?

1. Consider the Green function $G(x,y,k)$ of the exterior Dirichlet problem:

$$(\Delta + k^2)\, G = -\delta(x - y) \text{ in } \Omega \tag{1}$$

$$G|_{\Gamma} = 0 \tag{2}$$

$$|x|(\partial G/\partial |x| - ikG) \to 0 \text{ as } |x| \to \infty,\ k > 0. \tag{3}$$

Let $G_0 = (4\pi r_{xy})^{-1} \exp(ikr_{xy})$. Then

$$G(x,y,k) = G_0(x,y,k) - \int_\Gamma G_0(x,t,k)\mu(t,y,k)dt,\ \mu = \frac{\partial G}{\partial N_t}, \tag{4}$$

where N is the unit of the outer normal to Γ at the point t, and μ satisfies the equation

$$\mu + A\mu = 2\frac{\partial G_0}{\partial N}, \tag{5}$$

where A is defined by formula (1.6). Operator $A = A(k)$ is an entire function of k and $A(k)$ is compact in $H = L^2(\Gamma)$ for any k since Γ is smooth. It is invertible for $\text{Im}k > 0$. Hence, $(I + A(k))^{-1}$ is meromorphic and is defined on the whole complex plane k. Since $\partial G_0/\partial N$ for $y \notin \Gamma$ is an element of H which is an entire function of k, one can see from (5), that $\mu = 2(I + A(k))^{-1}\ \partial G_0/\partial N$ is meromorphic. From this argument and formula (4) it follows that $G(x,y,k)$ is medomorphic in k.

In §1 we emphasized that the location and properties of the complex poles of G are of interest in applications. By the properties of the poles we mean mostly whether the poles are simple or not.

Proposition 1. The set of the poles of G coincide with the set of the zeros of functions $\lambda_n(k)$, $n = 1, 2, 3\ldots$, where $\lambda_n(k)$ are the eigenvalues of the operator $T(k)$ defined by formula (1.9).

Proof. Let z be a pole of G,

$$G = \frac{R(x,y)}{(k-z)^r} + \ldots \tag{6}$$

From (4), (6), (2) after multiplying (4) by $(k-z)^r$ and taking $k = z$ we obtain

$$\int_\Gamma G_0(s,t,z)\,\frac{\partial R(t,y)}{\partial N_t}\,dt = 0, \qquad s\in\Gamma. \tag{7}$$

Since $R(x,y)$ is a degenerate kernel it follows from (7) that a function

$f(t) \not\equiv 0$ exists such that

$$\int_\Gamma G_0(s,t,z)\, f(t)dt = 0, \quad s\varepsilon\Gamma. \tag{8}$$

It means that $\lambda_n(z) = 0$ for some n.

Conversely, let equation (8) has a nontrivial solution. The function

$$u(x) = \int_\Gamma G_0(x,t,z)\, f(t)dt \tag{9}$$

is a solution of the exterior Dirichlet problem

$$(\Delta + z^2)\, u = 0 \text{ in } \Omega, \; u|_\Gamma = 0, \tag{10}$$

and u satisfies the asymptotic condition at infinity. (11)

If z is not a pole of G, $Imz \neq 0$, then $u \equiv 0$ in Ω and in D. It means that $f \equiv 0$ according to the jump relation. This is a contradiction. If z is not a pole of G and $Imz = 0$, then $u = 0$ in Ω and $u \neq 0$ in D only if z^2 is an eigenvalue of the interior Dirichlet problem for the Laplace operator. But such an eigenvalue is a (real) pole of G(x,y,k). Again, we obtain a contradiction. This completes the proof.

Remark 1. It is possible to find other functions whose zeros are poles of G [12].

Not much is known about the location of the complex poles of G:

1) It is proved in [33], [19] that the complex poles k_j of G (only Dirichlet boundary condition was considered) satisfy the following inequality:

$$Imk_j < a + b \ln |k_j|, \quad b > 0 \tag{12}$$

2. In [7] it was proved that a strip $-\varepsilon < Imk < 0$, $\varepsilon > 0$ is free of the poles of the resolvent kernel of the Schrödinger operator with a finite potential $q(x)\varepsilon C^1$ for the exterior Dirichlet problem. This result shows that there exists a function F(x) with the properties

(1.23) such that the complex poles of the resolvent kernel of the Schrödinger operator with $q(x) \varepsilon C_0^1$ satisfy inequality (1.22) for the exterior Dirichlet problem.

3. In [19] a study of the poles $k_j = i\sigma_j$, $\sigma_j < 0$ was carried out. It was proved that there exist infinitely many of such poles, and the number of poles with $|\sigma_j| < \sigma$ was estimated asymptotically for $\sigma \to \infty$.

4. The resolvent kernel of the Laplace operator of the exterior boundary value problem with the third boundary condition can have a pole $k = 0$. In this case the solution of the corresponding nonstationary problem for the wave equation does not necessarily decay as $t \to \infty$. An example is given in [34] where the problem

$$u_t = \Delta u \text{ in } \Omega = \{|x| \geq R,\ t \geq 0\} \tag{13}$$

$$u(x,0) = 0, u_t\ (x,0) = f(r), \tag{14}$$

$$\partial u/\partial r + R^{-1} u = 0, \text{ for } r = |x| = R,\ t \geq 0 \tag{15}$$

was considered. The solution can be found in the form

$$u = \sum_{n,m} u_{nm}\ (r,t)\ \gamma_{nm}(w), \tag{16}$$

where γ_{nm} are the spherical harmonics. From the explicit formula for u_{nm} it can be seen that $u_{00}(r, t)$ does not decay as $t \to \infty$ if $f(r) \geq 0$ and is finite. Another example is given in [32].

5. In [22] a criterion is given for an operator function $[I + A(k)]^{-1}$ to have only simple poles. If z is a pole of this function, $I + A(k) = I + A(z) + (k - z)\ \dot{A}_1 + \ldots$ then z is a simple pole iff

$$H = R\ (I + A(z)) \dot{+} A_1 \ker \{I + A(z)\}. \tag{17}$$

Unfortunately in order to apply this criterion in practice it is necessary to have such information about $I + A(z)$ and A_1, which is usually unavailable.

§5. How to calculate the poles of the Green function? Do the poles depend continuously on the boundary of the obstacle?

1. A general method for calculation the poles of Green functions in diffraction and scattering was given in [12], [13]. The poles coincide with the numbers k_j for which $I + A(k)$ is not invertible (see equation (4.5)). Let $\{f_j\}$ be an orthonormal basis in $H = L^2(\Gamma)$,

$$\mu_n = \sum_1^n c_j f_j. \tag{1}$$

Substituting (1) in (4.5) and multiplying in H by f_i one obtains the system for unknown c_j:

$$\sum_{i=1}^{n} b_{ij}(k) c_j = 0, \quad b_{ij} \equiv ([I + A(k)]f_j, f_i). \tag{2}$$

Here $(.,.)$ denotes the scalar product in H. System (2) has a non-trivial solution iff

$$\det [b_{ij}(k)] = 0. \tag{3}$$

The left-hand side of this equation is an entire function of k. Let $k_m^{(n)}$, m = 1, 2, 3,...be its roots. In [13] the following proposition is proved.

Proposition 1. The limits $\lim_{n\to\infty} k_m^{(n)} = k_m$ exist and are the poles of the Green function $G(x,y,k)$ of the exterior Dirichlet problem. Every pole of $G(x,y,k)$ can be obtained in such a way.

Remark 1. The same approach is valid for various boundary conditions (Neumann and third boundary conditions included), and for the potential scattering by a finite potential [12].

Remark 2. This approach is a variant of the general projection method.

Sketch of the proof. First we show that $k_m^{(n)} \to k_m$ as $n \to \infty$. In the complex plane we choose a circle K_R of arbitrary radius R. Suppose that the points $k_1, \ldots, k_s$ for which $I + A(k)$ is not invertible lie inside K_R and the remaining points k_m lie outside K_R. Denote by $\varepsilon > 0$ a small number, by $D_{\varepsilon,R} = \{k: \ |k - k_j| \geq \varepsilon, \ |k| \leq R\}$. We assume that the circles $|k - k_j| \leq \varepsilon$, $1 \leq j \leq s$ do not overlap. The operator $[I + A(k)]^{-1}$ is unfiromly bounded on $D_{\varepsilon,R}$:

$$\|[I + A(k)]^{-1}\| \leq M, \ k \varepsilon D_{\varepsilon,R}, \ M = M_{\varepsilon,R}. \tag{4}$$

Equation (2) can be written as

$$\mu_n + P_n A(k) \mu_n = 0, \tag{5}$$

where P_n is the projector on the span of $f_1, \ldots, f_n$. Since $P_n \to I$, where $\to$ denotes strong convergence of the operators on H, and A(k) is compact, we conclude that $\|A(k) - P_n A(k)\| \to 0$ as $n \to \infty$. Therefore $\|I + A(k) - [I + P_n A(k)]\| \to 0$ as $n \to \infty$. It means that for n sufficiently large operators $I + P_n A(k)$ are invertible in $D_{\varepsilon,R}$, because $I + A(k)$ is invertible in $D_{\varepsilon,R}$. Therefore all roots of equation (3) for n sufficiently large lie in the union of the circles

$$|k - k_j| \leq \varepsilon, \ |k| \leq R. \tag{6}$$

Since $\varepsilon > 0$ is arbitrarily small, this means that uniformly in the domain $|k| \leq R$ the limits exist:

$$\lim_{n\to\infty} k_j^{(n)} = k_j. \tag{7}$$

Conversely, let k_j, $|k_j| < R$ be an arbitrary pole of $G(x,y,k)$. Then operator $I + A(k_j)$ is not invertible. Suppose that in the circle

$|k_j - k| < \varepsilon$ there are no numbers $k_m^{(n)}$ and no points k_i for $i \neq j$. Then $\|[I + A(k)]^{-1}\| \leq M$ for $|k - k_j| = \varepsilon$ and for n sufficiently large $\|[I + P_n A(k)]^{-1}\| \leq M_1$. Since there are no numbers $k_m^{(n)}$ inside the circles $|k - k_j| < \varepsilon$, the operator $I + P_n A(k)$ is invertible for $|k - k_j| \leq \varepsilon$ and $[I + P_n A(k)]^{-1}$ is an analytic operator function for $|k - k_j| \leq \varepsilon$. From the maximum modulus principle we obtain a uniform (with respect to n) estimate $\|[I + P_n A(k)]^{-1}\| \leq M_1$ for $|k - k_j| \leq \varepsilon$. But from this estimate we conclude that the operator $[I + A(k)]^{-1}$ exists for $|k - k_j| \leq \varepsilon$, which is a contradiction. This completes the proof.

Remark 3. The method gives a uniform approximation to the complex poles in any compact domain of the complex plane k.

2. In this section we show that in any compact domain of the complex plane the complex poles depend continuously on the boundary in the following sense. Consider a parametrized equation of the boundary Γ

$$x_j = x_j\,(t_1,\ t_2),\ 1 \leq j \leq 3,\ 0 \leq t_1,\ t_2 \leq 1. \tag{8}$$

where $x_j \varepsilon C^2$.

Assume that a boundary Γ_ε obeys the following equation

$$x_j(\varepsilon) = x_j(t_1,t_2) + \varepsilon y_j(t_1,t_2),\ 1 \leq j \leq 3. \tag{9}$$

Where $y_j \varepsilon C^2$. Let $G(G_\varepsilon)$ be the Green function of the exterior Dirichlet problem in Ω, $\partial\Omega = \Gamma$ (Ω_ε, $\partial\Omega_\varepsilon = \Gamma_\varepsilon$). Let k_j ($k_j(\varepsilon)$) be the poles of $G(G_\varepsilon)$.

Proposition 2. If $\varepsilon \to 0$ then $k_j(\varepsilon) \to k_j$ uniformly for $|k_j| \leq R$, where $R > 0$ is an arbitrary large fixed number.

Proof. Denote by $\Delta = \{0 \leq t_1,t_2 \leq 1\}$. Then k_j, $|k_j| \leq R$ are the points of the complex plane k at which the operator $I + A(k)$

defined by formula (4.5) is not invertible. Operator $I + A(k,\varepsilon)$ is not invertible at the points $k_j(\varepsilon)$. Here the operator $A(k,\varepsilon)$ is the counterpart of $A(k)$ for Γ_ε. Both operators can be written in the form

$$A(k,\varepsilon) = \int_\Delta \frac{\partial G_0}{\partial N} \mu \quad J(t,\varepsilon)dt_1\, dt_2, \tag{10}$$

where $J(t,\varepsilon)dt_1\, dt_2$ is the element dt of the area of Γ_ε; for $\varepsilon = 0$ we obtain the operator $A(k)$. Since $x_j,\ y_j \varepsilon C^2$ the function $J(t,\varepsilon)$ is continuous (actually $J(t,\varepsilon)\varepsilon C^1$)

$$\lim J(t,\varepsilon) = J(t) \text{ as } \varepsilon \to 0. \tag{11}$$

Thus,

$$\|A(k,\varepsilon) - A(k)\| \to 0 \text{ as } \varepsilon \to 0,\ |k| \le R. \tag{12}$$

Now we can use the arguments given in the proof of Proposition 1. The role of n is played by ε. Consider the union K_δ of the circles $|k - k_j| \le \delta$, where $\delta > 0$ is an arbitrary small fixed number, $|k_j| < R, 1 \le j \le s$ and the circles do not overlap. By $D_{R,\delta}$ we denote $K_R \backslash K_\delta$, $K_R = \{k: |k| \le R\}$.

In $D_{R,\delta}$ operator $I + A(k)$ is invertible. Because of condition (12) for ε sufficiently small the operator $I + A(k,\varepsilon)$ is also invertible in $D_{R,\delta}$. This means that $k_j(\varepsilon)\varepsilon K_\delta$ for an ε sufficiently small. Since $\delta > 0$ is arbitrarily small the proof of Proposition 2 is complete.

Remark 4. It is possible to estimate $k_j(\varepsilon) - k_j$. In a general setting this type of perturbation theory was studied in [40], [41].

Appendix 1. Losses in open resonators [14].

Diffraction losses for the n-th mode in an open confocal resonators can be calculated by the formula

$$\alpha_n = 1 - |\lambda_n|^2, \quad n = 0, 1, 2\ldots, \tag{1}$$

where λ_n are the eigenvalues of the following operator

$$Af = \lambda f, \quad Af = \frac{b}{2\pi} \int_S \exp\{-ib(x,u)\} f(u)du, \tag{2}$$

and $S \subset \mathbb{R}^2$ is a central-symmetric domain, $b > 0$. It is easy to verify that A is normal. Thus $|\lambda_n| = s_n$, where $s_n = \lambda_n\{(A^*A)^{1/2}\}$ are the s-numbers of A. From the result, given in [36] it follows, that

$$s_n(S_1) \le s_n(S_2) \text{ if } S_1 \subset S_2. \tag{3}$$

From this we obtain the following inequalities

$$\alpha_{ne} \le \alpha_n \le \alpha_{ni}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where α_{ne} (α_{ni}) are the losses for the mirrors S_e (S_i), $S_e \supset S \supset S_i$. In [14] the following formula was also obtained:

$$|\lambda_n|^2 = \min_{L_n} \max_{\substack{f \perp L_n \\ \|f\| = 1}} \|Af\|^2, \quad n = 0, 1, 2, \ldots, \tag{5}$$

where L_n is a n-dimensional subspace of $H = L^2(S)$. The following conjecture was discussed in [14]: among all central-symmetric mirrors S with a fixed area $|S|$ the circle has minimal diffraction losses.

Appendix 2. An example on complex scaling.

In connection with spectral properties of the Schrödinger operator recently the complex scaling technique has attracted much attention [16]. The main idea is to consider solutions of the Schrödinger equation for complex values of $r = |x|$.

This idea was used by the author as early as 1963 in order to prove the absense of positive discrete spectrum of the Laplace operator of the Dirichlet problem in some infinite domains with infinite boundaries [15]. The arguments given in [15] are not elementary. Here we use the same idea as in [15] and give a very simple proof of the following (known) proposition.

Proposition 1. Let $D \subset R^3$ be a bounded domain with a smooth closed connected boundary Γ, $\Omega = R^3 \setminus D$,

$$(\Delta + k^2)u = 0 \text{ in } \Omega,\ k^2 > 0, \tag{1}$$

$$u \in L^2(\Omega), \tag{2}$$

$$u|_\Gamma = 0. \tag{3}$$

Then $u(x) \equiv 0$ in Ω.

Proof. By the Green formula we have

$$u(x) = -\int_\Gamma g^+ \mu dt,\ \mu = \frac{\partial u}{\partial N},\ g^+ = \frac{\exp(ikr_{xy})}{4\pi r_{xy}}. \tag{4}$$

(From (2) it follows, that $\nabla u \in L^2(\Omega)$ and hence a sequence $r_n \to \infty$ exists such that

$$\int_{|x|=r_n} \{|u|^2 + |\partial u/\partial N|^2\} ds \to 0 \text{ as } n \to \infty. \tag{5}$$

Therefore the integral over the large sphere in the Green formula tends to zero). Let $x = rw$, where w is a unit vector, and let $z = r\exp(i\theta)$. The function $u(x) = u(rw)$ is considered as a function of the complex variable z. Since

$$g^+ = \exp\frac{\{ik\sqrt{r^2 - 2r|t|\cos\alpha + |t|^2}\}}{\sqrt{r^2 - 2|t|r\cos\alpha + |t|^2}},\ \alpha = \hat{wt}, \tag{6}$$

it is clear, that G_0 is analytic in $z = r\exp(i\theta)$ for $|z| \geq R$, where R is sufficiently large, such that if $r > R$ then the inequality holds:

$$r^2 > 2rd + d^2,\ d = \max_{t \in \Gamma} |t|. \tag{7}$$

Thus for $|z| > R$ the function $\sqrt{z^2 - 2z|t|\cos\alpha + |t|^2}$ is analytic if we fix some branch of the radical. From (3) it follows that

$$u = \frac{\exp(ikz)}{z} f_1(z), \tag{8}$$

where f_1 is analytic in $|z| > R$ and

$$f_1 = 0(1) \text{ for } |z| > R. \tag{9}$$

Exactly the same arguments lead to the formulas:

$$u = -\int_\Gamma g^- \mu dt, \quad g^- = \frac{\exp(-ikr_{xy})}{4\pi r_{xy}}, \tag{10}$$

$$u = \frac{\exp(-ikz)}{z} f_2(z), \tag{11}$$

where $f_2(z)$ is analytic in $|z| > R$ and

$$f_2 = 0(1) \text{ for } |z| > R. \tag{12}$$

Hence

$$u(z) = \frac{e^{ikz}}{z} f_1 = \frac{e^{-ikz}}{z} f_2(z) \text{ for } |z| > R. \tag{13}$$

Formula (12) is contradictory unless $u \equiv 0$. To prove that we use a known uniqueness lemma for analytic functions.

Lemma. Let D be a domain on the complex plane z, C be its boundary. Let D contain the half plane $\mathrm{Re}z > a$. Let $f(z)$ be analytic in D, continuous in D + C and

$$\ln|f(z)| \le A|z| \text{ for } |z| > R, \ z\varepsilon D, \tag{14}$$

where $A = \text{const} > 0$, and R is an arbitrary large fixed number,

$$\ln|f(z)| \le -h(|z|), \ z\varepsilon C, \tag{15}$$

where $h(t) > 0$ is a continuous function such that

$$\int_1^\infty t^{-2} h(t)dt = \infty. \tag{16}$$

Then $f(z) \equiv 0$ in D.

In our case $f(z) = u(z)$, D can be chosen so that C coincides outside of some large circle with the rays $\arg z = 3\pi/4$, $\arg z = 5\pi/4$, $h(t) = \text{const} + \frac{t}{\sqrt{2}}$, so that (16) is satisfied. We have

$$\ln|u(z)| \le k|z| - \ln|z| + \ln|f_1| \le A|z|, \quad z\varepsilon D,$$

(since $|f_1| \le C_1$ we have $\ln|f_1| \le C_2$).

$$\ln|u(z)| \le -\frac{k|z|}{\sqrt{2}} + \text{const}, \text{ for } |z| > R, \quad z = |z|\exp(i3\pi/4).$$

Similar estimate holds on the ray $\arg z = 5\pi/4$. From the preceding lemma it follows that $u(z) \equiv 0$. Thus $u(r,w) = 0$ for $r > R$. By the unique continuation theorem we conclude that $u \equiv 0$ in Ω.

Appendix 3. Variational principles for eigenvalues of compact nonselfadjoint operators.

Let T be a linear compact operator on a Hilbert space, λ_j be its eigenvalues, $|\lambda_1| \ge |\lambda_2| \ge \dots$, $r_j(t_j)$ be the moduli of the real (imaginary) parts of the eigenvalues, $r_1 \ge r_2 \ge \dots (t_1 \ge t_2 \ge \dots)$. Let L_j, M_j, N_j be the eigenspaces of T corresponding to λ_j, r_j, t_j respectively. Note that r_j is not necessarily equal to $|\mathrm{Re}\lambda_j|$. We can set a one to one correspondence between r_j and $|\mathrm{Re}\lambda_j|$, and M_j and L_j, putting $M_i = L_{j(i)}$ where $j(i)$ is so chosen, that $|\mathrm{Re}\lambda_{j(i)}| = r_i$. The same is true for L_j and N_j. Let $\tilde{L}_j = \sum_{k=1}^{j} + L_k$ and $\tilde{M}_j$, $\tilde{N}_j$ are defined similarly.

Theorem. The following formulas hold:

$$|\lambda_j| = \max_{x\varepsilon\tilde{L}_{j-1}^{\perp}} \min_{\substack{y\varepsilon\tilde{L}_{j-1}^{\perp} \\ (x,y)=1}} |(Tx,y)| \tag{1}$$

$$r_j = \max_{x\in\tilde{M}^{\perp}_{j-1}} \min_{\substack{y\in\tilde{M}^{\perp}_{j-1}\\(x,y)=1}} |\mathrm{Re}(Tx,y)| \tag{2}$$

$$t_j = \max_{x\in\tilde{N}^{\perp}_{j-1}} \min_{\substack{y\in\tilde{N}^{\perp}_{j-1}\\(x,y)=1}} |\mathrm{Im}(Tx,y)|. \tag{3}$$

Here (x,y) denotes the scalar product in H.

Proof. We prove formula (1) for $j = 1$. The proof of other statements of the Theorem are similar. For $j = 1$ formula (1) can be written as

$$\lambda_1 = \max_{x\in H} \min_{\substack{y\in H\\(x,y)=1}} |(Tx,y)| \tag{4}$$

For a fixed x we write $Tx = \lambda x + z$, where $z\in x^{\perp}$, $x^{\perp}$ is the subspace of all vectors orthogonal to x and λ is a number. Thus $(Tx,y) = \lambda + (z,y)$. Let us represent y in the form $y = \mu x + u$, $u\in x^{\perp}$. From the condition $(x,y) = 1$ it follows that $\mu = |x|^{-2}$. Thus $(Tx,y) = \lambda + (z,u)$. We have

$$\min_{\substack{y\in H\\(x,y)=1}} |(Tx,y)| = \min_{u\in x^{\perp}} |\lambda + (z,u)| = \begin{cases} |\lambda| & \text{if } z = 0, \\ 0 & \text{if } z \neq 0. \end{cases} \tag{5}$$

Formula (5) implies (4).

Remark 1. If T_n is compact and $\|T - T_n\| \to 0$ as $n \to \infty$, then $\lambda_j(T_n) \to \lambda_j(T)$, $\forall_j$. This fact permits an approximate calculation of the spectrum of T using in (1)-(3) the operator T_n instead of T. In particular one can take n-dimensional operator T_n (dimrange $T_n = n$).

Bibliographical notes.

Section 1. Questions discussed here are of interest for engineers and physicists [2]-[4], [32]. They attracted considerable attention of mathematicians in recent years [1], [19], [3, Appendix]. Our knowledge of the spectral structure of nonselfadjoint operators is very limited. For example, it is not known how to investigate this structure of the equation (1.12). If a nonselfadjoint operator is a weak perturbation (in the sense defined in section 2) of a selfadjoint operator some information is available (see A. Marcus [17], V. Kacnelson [18], M. Agranovich [3]). For dissipative operators there exist some theorems about completeness of their root systems [1], [11]. No answer to question 5) is known.

Section 2. Properties of the bases of a Hilbert space are described in [1] in the form convenient for our purpose. A rigorous study of the spectral properties of the integral operators arising in diffraction theory was initiated in [12], [11], [13]. Questions put forward by B. Kacenelenbaum were stimulating for these studies. M. S. Agranovich [3] has made further contribution to this theory. Essential to his results were the results due to A. Markus [17] and V. Kacnelson [18]. The theory of pseudo-differential operators is now well developed. A summary of main results of this theory is given in [20], [21], [3], [38].

M. S. Agranovich [3] applied the theory of pseudo-differential operators to the integral equations of diffraction theory.

Section 3. References are given in the section.

Section 4. The questions discussed here are of interest in applications. Proposition 1 was proved in [29]. A part of it was proved in [12]. The scheme for the study of analytic continuation of the resolvent kernel of the Schrodinger operator was given in [5], [6], [25], [26]. Analytic properties of the scattering matrix for acoustic wave scattering by an obstacle was studied in [39]. Eigenfunction expansion theorems for nonselfadjoint Schrodinger operator are proved in [8], [9] and the properties of the resolvent in the complex plane of the spectral parameter k were used in the proofs. In [19] a study of the purely imaginary poles of the Green function of the exterior Dirichlet and the Neumann problem is given. The known criteria for a pole of an operator-valued function to be simple, including criterion (4.17) unfortunately are difficult to apply: so far no applications of these criteria appear to be known.

In [27] it is proved that for the complex poles of the Green function of the exterior Neumann problem for a convex domain in R^3 with a smooth boundary which has a positive Gaussian curvature, the function F(x) in formula (4.22) can be taken as $F(x) = \varepsilon|x|^{1/3}$, for some small $\varepsilon > 0$. In [28] it was shown how to pose correctly the problem of finding root vectors corresponding to the complex poles of the Green functions.

In [42] the analytic continuation of the resolvent of some general differential operators is studied.

There is an example in [37] which schows that a root system of a nonselfadjoint operator may form a basis of H, but some other root system of the same operator may not form a basis of H.

In the literature the radiation condition in the form $u \sim \frac{\exp(ikr)}{r}(1 + o(\frac{1}{r}))$ as $r \to \infty$ is often used for Imk < 0, i.e. for exponentially increasing solutions of the problem (4.10). It is assumed in such cases that the solution of the boundary value problem satisfying the radiation condition is unique. This is false. A simple example is the function $u = g^+ * f - g^- * f$, where g^+, g^- are defined by formulas (3), (9) of Appendix 2, $f \varepsilon C_0^\infty$ is arbitrary, * denotes convolution. It is clear that $(\Delta + k^2)\, u = 0$ in R^3 and u satisfies the radiation condition for $\mathrm{Im} k < 0$, but $u \not\equiv 0$. The right asymptotic condition for exponentially increasing solutions is given in [28], where it is proved that for $\mathrm{Im} z < 0$ the solution of the problem (4.10) has, in a neighborhood of infinity the following form $u = r^{-1} \exp(izr) \sum_{j=0}^{\infty} f_j(\alpha) r^{-j}$, $r = |x|$, $\alpha = x|x|^{-1}$, and the series converges absolutely and uniformly for sufficiently large r.

Section 5. The simple method for calculation of the complex poles is given in [12], [13]. It is essentially a variant of the projection method and the arguments show that the complex poles depend continuously on the boundary. The same arguments prove the continuous dependence of these poles on the parameters if the kernel depends continuously on these parameters.

The results of Appendix 3 was proved in [29]. In [30], [31] it was shown rigorously that the solution of the exterior Dirichlet boundary value problem is the limit of the solutions of the potential scattering problem when the potential goes to infinity in D and is equal to 0 in Ω. Here as usually $D = \mathbb{R}^3 \setminus \Omega$, Ω is the exterior domain. In [10] [35] behavior as $t \to \infty$ of the solution of the wave equation in exterior domain was studied in case when the resolvent kernel of the corresponding stationary problem cannot be analytically continued through the continuous spectrum.

It is possible to conclude from formula (12) in Appendix 2 that $u(z) \equiv 0$ without making use of Lemma of this Appendix. Indeed, since f_1, f_2 are analytic and bounded in some neighborhood of infinity they behave asymptotically as $C_n z^{-n}$, $n \geq 0$. If $z = iy$ in formula (12) of Appendix 2 and $y \to +\infty$, then the left-hand side of this formula goes to zero, while the right-hand side goes to infinity unless $f_1 = f_2 = 0$. This simple argument was pointed out by B.A. Taylor. In [15], where the boundary of the domain was infinite it was necessary to use Lemma from Appendix 2. It is interesting to mention that exactly the same arguments prove the following proposition.

Proposition 1. Let u be a solution of the problem (1)-(2) of Appendix 2. Then $u \equiv 0$.

Note that no assumptions about boundary values of u are made in this proposition.

Unsolved problems

1. To what extent do the complex poles of the Green function determine the obstacle?
2. Is it true that the complex pole of the Green function for the exterior Dirichlet problem are simple?
3. Does the order of a complex pole coincide with the order of zero of the corresponding eigenvalue? (see Proposition 1 in §4).

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