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Convergence of the Patch Zoning Method for Solving the MFIE

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ABSTRACT

A proof is given that, for a broad class of perfectly conducting scatterers, the patch zoning method for numerically solving the magnetic field integral equation provides a uniformly convergent sequence of approximate solutions for which the error can be bounded by an expression that tends to zero. The method of proof employs well-known techniques of abstract operator approximation theory. These techniques are applicable to the patch zoning method when a suitable Banach space is defined which includes both the class of solutions and the class of forcing functions appropriate to the electromagnetic scattering problem.

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SECTION 1. INTRODUCTION

The patch zoning method for numerically solving the magnetic field integral equation (MFIE) may be motivated by standard algorithms for performing numerical integration. The author's experience with this method has shown it to be a viable tool for predicting surface current densities on perfectly conducting metallic bodies illuminated by an electromagnetic field. The aim of this note is to show that, for a broad class of scattering surfaces, the patch zoning technique provides a sequence of approximate solutions which converge to the exact solution of the MFIE; details of the technique as applied to the MFIE may be found in Reference 1.

The proof of convergence will be based on the theory of collectively compact operator approximations [2]. We will cast the integral equation and the patch zoning approximations to that equation into the framework of this theory and thus establish the convergence of the patch zoning method. In addition, this abstract theory of operator approximation will permit us to place an upper bound on the magnitude of the difference between the approximate and the true solutions to the MFIE.

SECTION 2. PRELIMINARIES

For a perfectly conducting scattering body of the type to be considered in this paper, the magnetic field integral equation for a given wavenumber k takes the form

$$[(I-L)\vec{J}](\vec{r}) \equiv \vec{J}(\vec{r}) - \int_S \ell(\vec{r},\vec{r}\,')\vec{J}(r')dS' = 2\hat{n}(\vec{r}) \times \vec{H}^{inc}(\vec{r}) \qquad (1)$$

where $\hat{n}(\vec{r})$ is the unit outward normal at $\vec{r}$, $\vec{H}^{inc}(\vec{r})$ is the incident magnetic field, $\vec{J}(\vec{r})$ is the unknown tangential vector at $\vec{r}$ and

$$\ell(\vec{r},\vec{r}\,')\vec{J}(r') = \frac{e^{ik|\vec{r}-\vec{r}\,'|}(ik|\vec{r}-\vec{r}\,'|-1)\hat{n}(\vec{r}) \times [(\vec{r}-\vec{r}\,') \times \vec{J}(\vec{r}\,')]}{2\Pi|\vec{r}-\vec{r}\,'|^3} \qquad (2)$$

The proof of the convergence of the numerical solution of Equation (1) will be based on a number of assumptions about the types of bodies to be considered. We now list all the properties of a perfectly conducting body which will be explicitly used. These properties are satisfied by a large class of interesting bodies including continuously, twice differentiable surfaces of convex bodies.

Assumption 1: The surface of the body, S, is simply connected, bounded and has a finite surface area.

Assumption 2: $\hat{n}(\vec{r})$ is well defined at each point of S. In addition, for any two points $\vec{r}_1$ and $\vec{r}_2$ of S, we have that $\hat{n}(\vec{r}_1)-\hat{n}(\vec{r}_2) = |\vec{r}_1-\vec{r}_2|\vec{T}(\vec{r}_1,\vec{r}_2)$ where $\vec{T}$ is a continuous function of both its parameters except at $\vec{r}_2 = \vec{r}_1$. We will assume that $|\vec{T}|$ is bounded as $\vec{r}_2$ approaches $\vec{r}_1$.

Assumption 3: We can introduce an orthogonal coordinate system (e_1, e_2) on S such that their corresponding unit vectors $\hat{e}_1$

and $\hat{e}_2$ are well defined and continuous except at a finite number of points p_i. We will choose our orientation such that $\hat{e}_1 \times \hat{e}_2 = \hat{n}$.

Assumption 4: For each pair of points $\vec{r}_1$ and $\vec{r}_2$ of S with $\vec{r}_1 \neq \vec{r}_2$, there exists a point $\vec{r}_3 \neq \vec{r}_1$ on S with $|\vec{r}_1-\vec{r}_3| \leq |\vec{r}_1-\vec{r}_2|$ and $(\vec{r}_1-\vec{r}_2)$ parallel to the plane tangent to S at $\vec{r}_3$. In addition, given $\varepsilon>0$ there exists a $\delta>0$ such that for all $\vec{r}_4$ with $|\vec{r}_2-\vec{r}_4|<\delta$ there exists an $\vec{r}_5$ with the properties $|\vec{r}_5-\vec{r}_3|<\varepsilon, |\vec{r}_1-\vec{r}_5|<|\vec{r}_1-\vec{r}_4|$ and $|\vec{r}_1-\vec{r}_4|$ parallel to the plane tangent to S at $\vec{r}_5$.

In Appendix A we will use these assumptions to analyze the functional form of $\ell(\vec{r}, \vec{r}\,')\vec{J}(r')$. This in turn will be used in Appendix C to show that $(L\vec{J})(\vec{r})$ is continuous when $|\vec{J}|$ is bounded and $\int|\ell(\vec{r},\vec{r}\,')\vec{J}(\vec{r}\,')|dS'$ exists. Since $(I-L)\vec{J}$ would therefore have the same discontinuities as $\vec{J}$, the only bounded absolutely integrable solutions of $(I-L)\vec{J} = 2\hat{n} \times \vec{H}^{inc}$ for continuous $\vec{H}^{inc}$ must be continuous. We may therefore restrict the domain of L to C(S), the set of continuous tangential vector fields defined on the surface S. C(S) becomes a Banach space by introducing a uniform norm

$$||\vec{J}|| = \max_{\vec{r}} |\vec{J}| \tag{3}$$

This in turn induces a norm on the bounded operators of the Banach space; namely,

$$||K|| = \sup_{||\vec{J}||\leq 1} ||K\vec{J}|| \tag{4}$$

SECTION 3. PATCH ZONING

In Appendix A we will show that the integrand of our operator equation is of the form

$$\ell(\vec{r},\vec{r}\,')\vec{f}(\vec{r}\,') = \frac{\vec{v}(\vec{r},\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{5}$$

where $\vec{v}(\vec{r},\vec{r}\,')$ is a uniformly bounded function of its arguments. In addition, if $\vec{f}$ is piecewise continuous so is $\vec{v}$. With this in mind we will motivate the patch zoning approximation to the integral operator. In Appendix B we will show that this approximation is convergent.

A standard scheme for numerically calculating $I = \int_a^b f(e)de$ is to divide the interval [a,b] into m segments, q_i, and approximate I by

$$\sum_{i=1}^{m} f(e_i)\int_{q_i} de \tag{6}$$

where e is the midpoint of the i^{th} interval. If f(e) is of the form

$$f(e) = \frac{g(e)}{\left|e-\frac{a+b}{2}\right|^{1/2}} \tag{7}$$

we should modify Equation (6) to avoid the convergence difficulties imposed by the singularity. We choose a product integration scheme based on Riemann-Stieltjes integration to approximate the integral; namely

$$\int_a^b f(e)\,de = \int_b^a g(e)\,d\left[\mathrm{sgn}\left(e-\frac{a+b}{2}\right)\ \left|e-\frac{a+b}{2}\right|^{1/2}\right]$$

$$\approx \sum_i g(e_i) \int_{q_i} d\left[\mathrm{sgn}\left(e-\frac{a+b}{2}\right)\ \left|e-\frac{a+b}{2}\right|^{1/2}\right] \quad (8)$$

This concept has a natural extension to multiple integrals, and we use this extension to define our patch zoning method. We partition the surface into N zones, S_i, by an algorithm such that the diameter, $d_{i(N)}$, of each zone tends to zero as N increases. We then form the approximate operator $L_N J$ as follows

$$L_N\vec{J} \equiv \sum_{i=1}^{N} \int_{S_i} \left(\frac{\exp(ik|\vec{r}-\vec{r}\,'|)}{2\Pi|\vec{r}-\vec{r}\,'|^3} (ik|\vec{r}-\vec{r}|-1) \cdot \right.$$

$$\left. \hat{n}(\vec{r}) \times [(\vec{r}-\vec{r}\,') \times \tilde{J}(\vec{r}\,')] \right) dS' \quad (9)$$

where

$$\tilde{J}_i(\vec{r}\,') = [\vec{J}(\vec{r}_i)\cdot\hat{e}_1(\vec{r}_i)]\ \hat{e}_1(\vec{r}\,') + [\vec{J}(\vec{r}_i)\cdot\hat{e}_2(\vec{r}_i)]\hat{e}_2(\vec{r}\,') \quad (10)$$

and $\vec{r}_i$ is some central point of the i^{th} zone.

To minimize the number of discontinuities introduced by the patch zoning method, we will make all the singular points of the coordinate system lie on zone boundaries.

SECTION 4. CONVERGENCE OF THE SOLUTION

The approximations, L_N, to the magnetic field operator L were defined in the previous section. We will now state three abstract theorems about collectively compact approximations to an operator (Theorems 1.6, 1.9 and 1.11 of Reference 2) and use these theorems to show that these approximate operators define a sequence of approximate solutions to the MFIE which converge to the true continuous solution if one exists.

Definition: A sequence of operators K_N mapping a Banach space, X, into itself is called <u>collectively compact</u> if it maps the unit ball in X onto a relatively compact subspace of X.

Theorem 1: If K is compact, $\{K_N\}$ collectively compact, and $K_N \to K$, then $(I-K)^{-1}$ exists if, and only if, for some n, $(I-K_N)^{-1}$ exists and is bounded for all N>n. In that case $(I-K_N)^{-1} \to (I-K)^{-1}$.

Theorem 2: For K and K_N as in Theorem 1.

$$||(K_N-K)\ K_N|| \to 0 \tag{11}$$

Theorem 3: For K and K_N as in Theorem 1, we write $x = (I-K)^{-1}y$, $x_N = (I-K_N)^{-1}y$ and $\Delta_N = ||(I-K)^{-1}||\cdot||(K_N-K)K_N||$. Then for all N>n (same n as in Theorem 1) the error in the approximate solution is bounded. The bound is given by

$$||x_N - x|| \leq \frac{||(I-K)^{-1}||\left(||K_N y - Ky|| + \Delta_N ||y||\right)}{1-\Delta_N} \tag{12}$$

Note: The convergence of x_N to x follows from Theorem 1 and the existence of $(I-K)^{-1}$; Theorem 2 established that Δ_N tends to zero; and therefore, the assumption that $K_N \to K$ permits Theorem 3 to bound the error by an expression which tends to zero as N increases.

Since Appendices B and C show that the assumptions of Theorems 1 to 3 met by L and L_N, we have shown that the sequence of approximate solutions converges to the unique continuous solution if one exists. It remains to describe the method of solving the approximate equation. If we set

$$\vec{J}_N(\vec{r}) = 2\hat{n}(\vec{r}) \times \vec{H}^{inc}(\vec{r}) + (L_N \vec{J}_N)(\vec{r}) \tag{13}$$

then $\vec{J}_N(\vec{r})$ is completely determined by its values at the central points of the zones. We therefore restrict $\vec{r}$ to these points to obtain the matrix equation

$$\begin{pmatrix} I-A_N & -B_N \\ -C_N & I-D_N \end{pmatrix} \begin{pmatrix} \vec{J}_N \cdot \hat{e}_1 \\ \vec{J}_N \cdot \hat{e}_2 \end{pmatrix} (\vec{r}_i) = 2 \begin{pmatrix} -\hat{e}_2 \cdot \vec{H}^{inc} \\ \hat{e}_1 \cdot \vec{H}^{inc} \end{pmatrix} (\vec{r}_i) \tag{14}$$

where

$$\left.\begin{aligned} A_N &= \hat{e}_1(\vec{r}_i) \cdot (L_N \hat{e}_1)(\vec{r}_i) \\ B_N &= \hat{e}_1(\vec{r}_i) \cdot (L_N \hat{e}_2)(\vec{r}_i) \\ C_N &= \hat{e}_2(\vec{r}_i) \cdot (L_N \hat{e}_1)(\vec{r}_i) \\ D_N &= \hat{e}_2(\vec{r}_i) \cdot (L_N \hat{e}_2)(\vec{r}_i) \end{aligned}\right\} \tag{15}$$

The separation into surface coordinates is possible since we may restrict each $\vec{r}_i$ to $S-\{p_i\}$. Equation 14 is the scheme actually used for the numerical solution of the MFIE via patch zoning and, by construction, its solution is identical to the value at each of the $\vec{r}_i$ of the solution to the approximate integral equation. The maximum error on the set $\{\vec{r}_i\}$ is certainly no greater than the maximum error on the entire surface, thus leading to the conclusion that the numerical solution converges along with the approximate solution.

APPENDIX A. BEHAVIOR OF $\ell(\vec{r},\vec{r}\,')$

In this appendix we will discuss the nature of the discontinuities of the integrand of the magnetic field operator L. Our intention is to prove enough to show that L maps uniformly bounded, integrable, tangential vector fields into equicontinuous tangential vector fields. This property of L will be used to show that L is compact and that $\{L_N\}$ is collectively compact.

We begin by showing that $\ell(\vec{r},\vec{r}\,')\vec{J}(\vec{r}\,')$ can be written as

$$\ell(\vec{r},\vec{r}\,')\vec{J}(\vec{r}\,') = \frac{\vec{v}(\vec{r},\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad \text{for } \vec{r}\,' \neq \vec{r} \qquad \text{(A-1)}$$

where $\vec{v}$ is a bounded vector field, necessarily tangential at $\vec{r}$, defined everywhere except $\vec{r}\,' = \vec{r}$. The continuity parts of assumptions 2 and 4 will guarantee that for fixed $\vec{r}\,'$, $\vec{v}$ is continuous in $\vec{r}$. In addition, for fixed $\vec{r}$, $\vec{v}$ will be continuous at $\vec{r}\,'$ if $\vec{J}$ is continuous at r'. To do this we must evaluate $\vec{g}(\vec{r},\vec{r}\,')$ where $\vec{g}(\vec{r},\vec{r}\,') \equiv \hat{n}(\vec{r}\,') \times \left[(\vec{r}-\vec{r}\,') \times \vec{J}(r')\right]$. This triple product can be expanded to give

$$\vec{g}(\vec{r},\vec{r}\,') = [\hat{n}(\vec{r})\cdot\vec{J}(\vec{r}\,')](\vec{r}-\vec{r}\,') - [\hat{n}(\vec{r})\cdot(\vec{r}-\vec{r}\,')]\vec{J}(r') \qquad \text{(A-2)}$$

From assumptions 2 and 4, and from the fact that $\hat{n}(r')\cdot J(r')=0$ we have

$$[\hat{n}(\vec{r})\cdot\vec{J}(\vec{r}\,')](\vec{r}-\vec{r}\,') = ([\hat{n}(\vec{r})-\hat{n}(\vec{r}\,')]\cdot\vec{J}(\vec{r}\,'))\,|\vec{r}-\vec{r}\,'|\,\hat{t}(\vec{r}\,'')$$

$$= |\vec{r}-\vec{r}\,'|^2\left[\vec{T}(\vec{r},\vec{r}\,')\cdot J(\vec{r}\,')\right]\hat{t}(\vec{r}\,'') \qquad \text{(A-3)}$$

But $\vec{T}$ is continuous in $\vec{r}$ and $\vec{r}'$ as is t, provided that $\vec{r}' \neq \vec{r}$; hence, the result is continuous in $\vec{r}$. And, if $\vec{J}$ is continuous at $\vec{r}'$, so is the resultant first term of the right-hand side of Equation A-2.

Similarly, we have for the second term

$$[\hat{n}(\vec{r}) \cdot (\vec{r}-\vec{r}')]\vec{J}(\vec{r}') = |\vec{r}-\vec{r}'| \Big([\hat{n}(\vec{r})-\hat{n}(\vec{r}'')] \cdot \hat{t}(\vec{r}'') \Big) \vec{J}(\vec{r}')$$

$$= \frac{|\vec{r}-\vec{r}''|}{|\vec{r}-\vec{r}'|} |\vec{r}-\vec{r}'|^{\hat{2}} [\vec{T}(\vec{r},\vec{r}'') \cdot \hat{t}(\vec{r}'')]\vec{J}(r') \qquad \text{(A-4)}$$

where, by assumption 4, the leading coefficient of the result is less than or equal to unity. The continuity with respect to $\vec{r}$ and $\vec{r}'$ follows exactly as with Equation A-3.

Combining Equations A-3 and A-4 and their ensuing discussion, we find that we have proved our assertions about $\vec{v}$. Since the continuity of $\vec{v}$ with respect to $\vec{r}$ was independent of the continuity of $\vec{J}$, the above analysis of that continuity is equally valid for $L_N\vec{J}$. In particular we have shown that

$$|\vec{v}(\vec{r}',\vec{r}'')| \leq \frac{1+kd(S)}{\Pi} (\max|\vec{T}|) \; ||\vec{J}|| \qquad \text{(A-5)}$$

and therefore by assumption 1 $\int \frac{\vec{v}(\vec{r},\vec{r}')}{|\vec{r}-\vec{r}'|} dS'$ exists.

APPENDIX B. CONVERGENCE OF $L_N\vec{J}$ to $L\vec{J}$

In this appendix we prove that for any element $\vec{J}$ of C(S), $||(L-L_N)\vec{J}||$ approaches zero as N approaches infinity; that is, given $\varepsilon>0$ $\exists N_0$ such that $\forall N>N_0$ $||(L-L_N)J||<\varepsilon$. Our procedure will be to divide the surface S, into disjoint surfaces S_ε and O_ε, where O_ε is the union of open discs about the p_i's. We then estimate the magnitude of the integral of $|(L-L_N)\vec{J}|$ over each surface.

Recalling the rule from analysis that the magnitude of a sum is less than or equal to the sum of the magnitudes, we have that

$$||(L-L_N)\vec{J}|| \leq \max_{\vec{r}} \sum_i \int_{S_i \cap S_\varepsilon} |\ell(\vec{r},\vec{r}\,')\left[\vec{J}(\vec{r}\,')-\tilde{J}_i(\vec{r}\,')\right]|dS'$$

$$+ \sum_i \int_{S_i \cap O_\varepsilon} |\ell(\vec{r},\vec{r}\,')\left[\vec{J}(\vec{r}\,')-\tilde{J}_i(\vec{r}\,')\right]|dS' \qquad \text{(B-1)}$$

We now proceed to show that each of the sums on the right-hand side of equation B-1 may be bounded by $\varepsilon/2$.

One can show, via the triangle inequality, that

$$|\vec{J}(\vec{r}\,')-\tilde{J}_i(\vec{r}\,')|\leq|\vec{J}(\vec{r}\,')|+|\tilde{J}(\vec{r}\,')|\leq 2||\vec{J}|| \qquad \text{(B-2)}$$

The estimate of Equation A-5 then shows that it is possible to choose O_ε so small that, independent of $\vec{r}$, the magnitude of the second sum in Equation B-1 is less than $\varepsilon/2$.

We now show that it is possible to pick N_0 so large that the first sum in Equation B-1 is also bounded by $\varepsilon/2$. By our construction of S_ε we have that $\vec{r}'$ is bounded away from $\{p_i\}$. This guarantees that $\vec{J}$, $\hat{e}_1$ and $\hat{e}_2$ are uniformly continuous on S_ε, and hence for fixed η it is possible to choose $d_{i(N)}$ so small (N so large) that $|\vec{J}(\vec{r}')-\vec{J}(\vec{r}_i)|<\eta$, and $|\tilde{J}_i(\vec{r}')-\vec{J}(\vec{r}_i)|\leq\eta$ for $\vec{r}'$ in $S_\varepsilon \cap S_i$. But

$$|\vec{J}(\vec{r}) - \tilde{J}_i(\vec{r})| \leq |\vec{J}(\vec{r}) - \vec{J}(\vec{r}_i)| + |\tilde{J}_i(\vec{r}) - \vec{J}(\vec{r}_i)| \leq 2\eta \quad \text{(B-3)}$$

Hence, by Equation A-5 it is possible to set N_0 sufficiently large to guarantee the required bound for the first sum of Equation B-1, and we have proved what we set out to prove.

APPENDIX C. PROOF THAT $\{L_N\}$ IS COLLECTIVELY COMPACT

In this appendix we prove that $\{L_N\}$ is collectively compact. By the definition of collectively compact we must show that $\{L_N\}$ maps the unit ball, B, of our Banach space, C(S), onto a relatively compact subset of C(S). We will use the Arzela Ascoli Theorem to show that $\{L_N B\}$ is relatively compact.

We recall from Appendix A that $\ell(\vec{r},\vec{r}\,')\vec{J}(\vec{r}\,') = \vec{v}(\vec{r},\vec{r}\,')/|\vec{r}-\vec{r}\,'|$ where $\vec{v}$ is bounded and is continuous as a function of $\vec{r}$ over any set excluding $\vec{r}\,'$. The modulus of continuity is proportional to $||\vec{J}||$ and on compact subsets of S can be bounded without regard to any characteristics of $\vec{J}$ other than its maximum magnitude. With this in mind, we proceed to show that $\{L_N B\}$ is equicontinuous.

Consider $|(L_N\vec{J})(\vec{r}_1) - (L_N\vec{J})(\vec{r}_2)|$. We have

$$|(L_N\vec{J}(\vec{r}_1)-L_N\vec{J}(\vec{r}_2)| = \left|\int_S\left(\frac{\vec{v}(\vec{r}_1,\vec{r}\,')}{|\vec{r}_1-\vec{r}\,'|} - \frac{\vec{v}(\vec{r}_2,\vec{r}\,')}{|\vec{r}_2-\vec{r}\,'|}\right)dS'\right| \tag{C-1}$$

To prove that its magnitude is less than any preassigned ε for sufficiently close $\vec{r}_1$ and $\vec{r}_2$, we partition S into disjoint sets O_ε and S_ε where O_ε is an open disc centered at $\vec{r}_1$. We choose O_ε to be small enough to guarantee that

$$\int_{O_\varepsilon}\left(\frac{|\vec{v}(\vec{r}_1,\vec{r}\,')|}{|\vec{r}_1-\vec{r}\,'|} + \frac{|\vec{v}(\vec{r}_2,\vec{r}\,')|}{|\vec{r}_2-\vec{r}\,'|}\right)dS' \quad <\varepsilon/3 \tag{C-2}$$

This will always be possible since $\vec{v}$ is a bounded function of its parameters.

It remains to show that the integral over S_ε can also be made less than $2\varepsilon/3$.

$$\left| \int_{S_\varepsilon} \left(\frac{\vec{v}(\vec{r}_1,\vec{r}\,')}{|\vec{r}_1-\vec{r}\,'|} - \frac{\vec{v}(\vec{r}_2,\vec{r}\,')}{|\vec{r}_2-\vec{r}\,'|} \right) dS' \right| \tag{C-3a}$$

$$= \left| \int_{S_\varepsilon} \left(\frac{(|\vec{r}_2-\vec{r}\,'|-|\vec{r}_1-\vec{r}\,'|)\vec{v}(\vec{r}_1,\vec{r}\,')}{(|\vec{r}_2-\vec{r}|)(|\vec{r}_1-\vec{r}\,'|)} + \frac{\vec{v}(\vec{r}_1,\vec{r}\,')-\vec{v}(\vec{r}_2,\vec{r}\,')}{|\vec{r}_2-\vec{r}\,'|} \right) dS' \right| \tag{C-3b}$$

$$\leq \int_{S_\varepsilon} \frac{|\vec{r}_2-\vec{r}_1|\,|\vec{v}(\vec{r}_1,\vec{r}\,')|}{\delta_1\delta_2} dS' + \int_{S_\varepsilon} \frac{|\vec{v}(\vec{r}_1,\vec{r}\,')-\vec{v}(\vec{r}_2,\vec{r}\,')|}{\delta_2} dS' \tag{C-3c}$$

where δ is the minimum distance from $\vec{r}_1$ to S_ε, and δ_2 is the minimum distance from $\vec{r}_2$ to S_ε. If we assume that $\vec{r}_2$ is contained in O_ε, then δ_1 and δ_2 are nonzero. For sufficiently small $\vec{r}_2-\vec{r}_1$, the first integral of C-3 can be made less than $\varepsilon/3$ by virtue of the factor $|\vec{r}_1-\vec{r}_2|$. The second integral can also be made less than $\varepsilon/3$ by virtue of the continuity of $\vec{v}$ with respect to its first parameter.

We have therefore shown that $L_N J$ is uniformly continuous in a manner which does not depend on N or on any characteristic of $\vec{J}$ other than $||\vec{J}||\leq 1$. By the Arzela Ascoli Theorem, a uniformly bounded, equicontinuous set of functions is relatively compact with respect to the Banach space of continuous functions and hence $\{L_N\}$ is collectively compact. Since our proof is equally valid for L, we find that L is compact.

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