Mathematics Note

Note 57

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Evaluation of the Integral of the Anger-Weber Function with a Complex Argument

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Abstract

A series expression for small complex arguments and an asymptotic expression for large complex arguments has been derived for evaluating the integral of Anger-Weber functions. A study of the integral has been conducted for various values of order and complex argument.

INTRODUCTION

The Anger and the Weber functions[1] of the order ν and complex argument K are defined in terms of the following integral representations

$$\mathbf{J}_\nu(K) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - K\sin\theta)\, d\theta \qquad (1)$$

$$\mathbf{E}_\nu(K) = \frac{1}{\pi} \int_0^\pi \sin(\nu\theta - K\sin\theta)\, d\theta \qquad (2)$$

The Anger function $\mathbf{J}_\nu(K)$ reduces to the Bessel function $J_n(K)$ when $\nu = n$ is an integer, and the Weber function $\mathbf{E}_\nu(K)$ is same as the Lommel-Weber function $\Omega_\nu(K)$ with the change of sign,[2]

$$J_n(K) = \mathbf{J}_\nu(K), \quad \nu = n \text{ an integer} \qquad (3)$$

$$\Omega_\nu(K) = -\mathbf{E}_\nu(K) \qquad (4)$$

The integral of the particular linear combination of these two functions which has been called the Anger-Weber function,[3] is given by

$$V_\nu(K) = \frac{j}{2} \int_0^{2K} [\mathbf{J}_\nu(\zeta) + j\mathbf{E}_\nu(\zeta)]\, d\zeta \qquad (5)$$

which can be rewritten in the following form for ν an integer by substituting the expressions (3) and (4),

$$V_\nu(K) = U_n(K), \quad \nu = n \text{ an integer} \qquad (6)$$

$$U_n(K) = \frac{1}{2} \int_0^{2K} [\Omega_n(\zeta) + jJ_n(\zeta)]\, d\zeta \qquad (7)$$

where $J_n(\zeta)$ and $\Omega_n(\zeta)$ are the Bessel function and Lommel-Weber function respectively, of integer order n and complex argument ζ. The integral (7) is frequently used in the analysis of Torus type EMP simulators and loop antennas.

In the following sections, a series expansion for small complex arguments and an asymptotic expression for large complex arguments is derived for evaluating the integral expression given in (7). Using these results, the integral of the Anger-Weber function is tabulated for various values of order and complex argument. The integral is analytic in the finite complex plane K and one can introduce the general normalized transformation

$$
\begin{aligned}
S &= jK \\
&= \gamma a \qquad (8)
\end{aligned}
$$

to redefine the integral (7) in a convenient form suitable for EMP work. In the expression (8), a is an appropriate characteristic dimension which makes S dimensionless. Thus the integral (7) in the S plane is

$$
\begin{aligned}
W_n(S) &= U(-jS) \\
&= \frac{1}{2}\int_0^{-j2S} [\Omega_n(\zeta) + jJ_n(\zeta)]\, d\zeta \qquad (9)
\end{aligned}
$$

I. Series Expansion for Small Arguments

Combining the Bessel and the Lommel-Weber functions, we can write the bracketed term in (9) as

$$\Omega_n(\zeta) + jJ_n(\zeta) = \frac{j}{\pi}\int_0^{\pi} e^{-j[\zeta\sin\theta - n\theta]}\, d\theta \qquad (10)$$

Substituting the expression (10) into (9), with the transformation $p = j\zeta$,

$$W_n(S) = \frac{1}{2\pi}\int_0^{2S}\int_0^{\pi} e^{jn\theta}\, e^{-p\sin\theta}\, d\theta\, dp \qquad (11)$$

Performing the outside integration first,

$$W_n(S) = -\frac{1}{2\pi}\int_0^{\pi} e^{jn\theta}\left[\frac{e^{-2S\sin\theta} - 1}{\sin\theta}\right] d\theta \qquad (12)$$

Next, expanding the exponential of the bracketed term in a Taylor series, we obtain

$$W_n(S) = -\frac{1}{2\pi}\int_0^{\pi} e^{jn\theta}\left[\sum_{\ell=1}^{\infty} \frac{(-2S)^{\ell}(\sin\theta)^{\ell-1}}{\ell!}\right] d\theta \qquad (13)$$

Interchanging the summation and integration in the expression (13), one obtains,

$$W_n(S) = -\frac{1}{2\pi}\sum_{\ell=0}^{\infty} \frac{(-2S)^{\ell+1}}{(\ell+1)!}\int_0^{\pi} e^{jn\theta}(\sin\theta)^{\ell}\, d\theta \qquad (14)$$

where the integral part in the above expression (14) can be evaluated as,[5]

$$\int_0^{\pi} e^{jn\theta}(\sin\theta)^{\ell-1}\, d\theta = \frac{\pi\, e^{jn\pi/2}}{2^{\ell-1}\,\ell\,\beta\left[\frac{\ell+n+1}{2},\, \frac{\ell-n+1}{2}\right]} \qquad (15)$$

where $\beta(x,y)$ is defined in terms of the gamma function,[5]

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(y,x) \tag{16}$$

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \tag{17}$$

Substituting these results in the expression (15), we obtain

$$\int_0^{\pi} e^{jn\theta}(\sin\theta)^{\ell}\, d\theta = \frac{\pi(\ell!)\, e^{jn\pi/2}}{2^{\ell}\,\Gamma\left(\frac{\ell+n}{2}+1\right)\Gamma\left(\frac{\ell-n}{2}+1\right)} \tag{18}$$

where we have put $\Gamma(\ell+1) = \ell!$, ℓ an integer, and using the expression (18), the expression $W_n(S)$ for small arguments takes the form

$$W_n(S) = \sum_{\ell=0}^{\infty} \frac{j^n(-1)^{\ell}}{(\ell+1)} \frac{S^{\ell+1}}{\Gamma\left(\frac{\ell+n}{2}+1\right)\Gamma\left(\frac{\ell-n}{2}-1\right)} \tag{19}$$

The expression $W_n(S)$ is well behaved in the complex S plane. Also it has the property $W_n(S^*) = [W_n(S)]^*$. In fact the expression $W_n(S)$ is real for all values of real S and n an even integer.

The series may be computed directly, however, the accuracy is limited for large values of $|S|$. In this study, the series (19) was summed numerically to obtain values of $W_n(S)$. The following criterion for the series expression (19) is used for convergence check on a CDC 6600,

$$\left|\frac{W_n^L - W_n^{L-1}}{W_n^L}\right| < 10^{-7} \tag{20}$$

where the superscript L denotes the partial sum of the series (19) with L terms.

It is noted that the expression (19) is same as that obtained in reference 3 with the substitution $S = jK$.

II. Asymptotic Expansion for Large Arguments

According to the expressions (5) and (6), we can write $W_n(S)$ as,

$$W_n(S) = \frac{j}{2} \int_0^{-2jS} \left[J_n(\zeta) + jE_n(\zeta) \right] d\zeta \tag{21}$$

where for the upper limit of the integration K = -jS is substituted from the definition (8). Further introducing the transformation $p = j\zeta$ for the integration variable, the expression (21) can be rewritten as,

$$W_n(S) = \frac{1}{2} \int_0^{2S} \left[J_n(\zeta) + jE_n(\zeta) \right]_{\zeta = -jp} dp \tag{22}$$

The asymptotic forms of the intregrand in (22) are found in reference 1, in terms of the Hankel function,

$$J_n(\zeta) + jE_n(\zeta) \sim H_n^{(2)}(\zeta) - j\left[\frac{1 + (-1)^n}{\pi\zeta}\right] + 0\left[\frac{1}{\zeta^2}\right] \tag{23}$$

$$\sim \left[\frac{2}{\pi\zeta}\right]^{\frac{1}{2}} e^{-j(\zeta - \frac{n\pi}{2} - \frac{\pi}{4})} - j\left[\frac{1 + (-1)^n}{\pi\zeta}\right],$$

$$|\arg \zeta| < \pi \tag{24}$$

Because of the presence of the term $1/\zeta$, the integral expression $W_n(S)$ is not convergent for n even but is convergent for n odd which can be shown by comparing $W_n(S)$ to the Fresnel integrals.[1] However, with the following rearrangement, where we have $\zeta = -jp$,

$$W_n(S) = \frac{1}{2} \int_0^{j1} \left[J_n(\zeta) + jE_n(\zeta) \right] dp + I_1(S) + I_2(S) \tag{25}$$

where

$$I_1(S) = \frac{1}{2}\int_{j1}^{2S} \frac{1+(-1)^n}{\pi}\,dp$$

$$= \frac{1+(-1)^n}{2\pi}\,\ln(-j2S) \tag{26}$$

and

$$I_2(S) = \frac{1}{2}\int_{j1}^{2S}\left[J_n(\zeta) + jE_n(\zeta) + j\,\frac{1+(-1)^n}{\pi\zeta}\right]dp \tag{27}$$

The integral $I_2(S)$ is convergent as $S \to \infty$, and hence can be rewritten in the form

$$I_2(S) = \frac{1}{2}\left[\int_{j1}^{j\infty} + \int_{j\infty}^{2S}\right]\left[J_n(\zeta) + jE_n(\zeta) + j\,\frac{1+(-1)^n}{\pi\zeta}\right]dp \tag{28}$$

For large S, the integrand in the second integral of (28) can be replaced by its asymptotic form to obtain,[7]

$$I(S) = \frac{1}{2}\int_{j\infty}^{2S}\left[J_n(\zeta) + jE_n(\zeta) + j\,\frac{1+(-1)^n}{\pi\zeta}\right]dp \tag{29}$$

$$\sim \frac{1}{2}\int_{j\infty}^{2S}\left[\left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} e^{-j(\zeta-n\pi/2-\pi/4)}\right]dp \tag{30}$$

$$|\arg(\zeta = -jp)| < \pi$$

The integral (30) simplifies to Fresnel integrals, if we now let $\frac{\pi}{2}t^2 = \zeta$, $u = -jS$,

$$I(S) \sim j\,e^{j(n\pi/2+\pi/4)}\int_{\infty}^{2(\frac{u}{\pi})^{\frac{1}{2}}} e^{-j\frac{\pi}{2}t^2}\,dt \tag{31}$$

$$= j\,e^{j(n\pi/2+\pi/4)}\left[\int_{\infty}^{0} + \int_{0}^{2(\frac{u}{\pi})^{\frac{1}{2}}}\right] e^{-j\frac{\pi}{2}t^2}\,dt \tag{32}$$

$$= \left[-\frac{1}{2} - j\frac{1}{2} + Si\left[2(\tfrac{u}{\pi})^{\frac{1}{2}}\right] + jCi\left[2(\tfrac{u}{\pi})^{\frac{1}{2}}\right]\right] e^{j(n\pi/2+\pi/4)}$$

$$|\arg u| < \pi \tag{33}$$

where the Fresnel integrals

$$\mathrm{Ci}(u) = \int_0^u \cos\left(\frac{\pi}{2} t^2\right) dt \tag{34}$$

$$\mathrm{Si}(u) = \int_0^u \sin\left(\frac{\pi}{2} t^2\right) dt \tag{35}$$

have the following asymptotic forms,[8] for $|\arg u| < \pi/2$,

$$\mathrm{Ci}(u) \sim \frac{1}{2} + \frac{1}{\pi u} \sin\left(\frac{\pi}{2} u^2\right) + O\left(\frac{1}{u^2}\right) \tag{36}$$

$$\mathrm{Si}(u) \sim \frac{1}{2} - \frac{1}{\pi u} \cos\left(\frac{\pi}{2} u^2\right) + O\left(\frac{1}{u^2}\right) \tag{37}$$

On substituting the expressions (36) and (37) into (33), we have

$$I(S) \sim \frac{1}{2(\pi u)^{\frac{1}{2}}} e^{-j[2u-n\pi/2-\pi/4]} \qquad |\arg u| < \pi/2 \tag{38}$$

Using the expressions (26), (28), and (38), for large values of S, the expression (21) takes the form, $u = -jS$,

$$W_n(S) \sim C_n + \frac{1+(-1)^n}{2\pi} \ln(-jS) - \frac{1}{2(\pi S)^{\frac{1}{2}}} e^{-2S+j(n+1)\pi/2} \qquad |\arg S| < \pi \tag{39}$$

where the coefficient

$$C_n = \frac{1}{2} \int_0^{j1} \left[J_n(\zeta) + jE_n(\zeta)\right] dp + \frac{1}{2} \int_{j1}^{j\infty} \left[J_n(\zeta) + jE_n(\zeta) + j\,\frac{1+(-1)^n}{\pi\zeta}\right] dp + \frac{1+(-1)^n}{2\pi} \ln 2 \tag{40}$$

To evaluate the constant C_n, we can make use of the already-derived asymptotic formula for $W_n(x)$ where x is real argument,[3]

$$C_n = \lim_{x\to\infty} \left[W_n(x) - \frac{1 + (-1)^n}{2\pi} \ln x\right] \tag{41}$$

Or one may follow the following method to evaluate the constant coefficient C_n. According to the expression (39),

$$C_n = \lim_{S\to\infty} \left[W_n(S) - \frac{1 + (-1)^n}{2\pi} \ln(-jS)\right] \tag{42}$$

Hence

$$C_{n-2} = \lim_{S\to\infty} \left[W_{n-2}(S) - \frac{1 + (-1)^{n-2}}{2\pi} \ln(-jS)\right] \tag{43}$$

Subtracting (43) from the expression (42),

$$C_n - C_{n-2} = \lim_{S\to\infty} \left[W_n(S) - W_{n-2}(S)\right] \tag{44}$$

Further we have from the expression (11),

$$W_n(S) = \frac{j}{2} \int_0^{-j2S} \left[\frac{1}{\pi} \int_0^{\pi} e^{jn\theta} e^{-j\zeta\sin\theta} d\theta\right] d\zeta$$

and hence the right-hand side of (44) reduces to

$$W_{n+1}(S) - W_{n-1}(S) = -j\left[J_n(-j2S) + jE_n(-j2S)\right] - \frac{1 - (-1)^n}{n\pi} \tag{45}$$

with n = n-1 substituted into (45), and in the limit as $S\to\infty$,

$$C_n - C_{n-2} = - \frac{1 - (-1)^{n-1}}{(n-1)\pi} \tag{46}$$

forms the recurrence relationship for C_n coefficients. Rearranging the C_n integral terms appropriately and noting that,[8]

$$\int_0^\infty J_n(\zeta)d\zeta = 1 \tag{47}$$

$$C_n = \frac{j}{2} - \lim_{S\to 0} \frac{1}{2} \int_0^{-jS} E_n(\zeta)d\zeta + \frac{1 + (-1)^n}{2\pi} \ln(-jS) \tag{48}$$

which gives for n = 0,[8]

$$C_0 = \frac{j}{2} - \frac{1}{\pi} \psi(\tfrac{1}{2})$$

and for n = 1, (49)

$$C_1 = \frac{j}{2}$$

Using C_0 and C_1, the recurrence relation (46) yields the general term,

$$C_n = \frac{j}{2} - \frac{1 + (-1)^n}{2\pi} \psi(\frac{m + 1}{2}) \tag{50}$$

where

$$\psi(\tfrac{1}{2}) = - \Gamma - 2 \ln 2$$

$$\psi(m+1) = \psi(m) + \frac{1}{m}$$

$\Gamma = 0.5772...$ Euler's constant

Hence, substituting the expression (50) for constant coefficient into the expression (39), for large values of S,

$$W_n(S) \sim \frac{j}{2} + \left[\frac{1 + (-1)^n}{2\pi}\right]\left[\ln(-jS) - \psi(\frac{m+1}{2})\right]$$

$$- \frac{1}{2(\pi S)^{\frac{1}{2}}} e^{-2S+j(n+1)\pi/2}$$

$$|\arg S| < \pi \tag{51}$$

III. Numerical Results and Applications

Using double precision and the series convergence criterion given in the equation (20), $W_n(S)$ may be evaluated accurately for $|S| < 30$ on the CDC 6600. A more detailed discussion on the range limitations for imaginary arguments may be found in Mathematics Note 25.[3]

Appendix A contains tables for $W_n(S)$ for several values of order n and complex argument S, while Appendix B gives the Fortran program listing used to evaluate $W_n(S)$.

Extensive application of the integral of the Anger-Weber function with complex argument can be found in references 6 and 9, in which the Singularity Expansion Method is applied to the study of circular loop antennas.

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APPENDIX A

$W_n(S)$ for Several Values of Order n and Complex Argument S

n	COMPLEX ARGUMENT S		INTEGRAL	$W_n(S)$
0	0.00000	0.00000	0.00000	0.00000
0	0.00000	.50000	.15055	.45987
0	0.00000	1.00000	.50935	.71289
0	0.00000	1.50000	.86639	.69378
0	0.00000	2.00000	1.04577	.51237
0	0.00000	2.50000	1.02212	.35766
0	0.00000	3.00000	.91574	.35311
0	0.00000	3.50000	.87990	.47732
0	0.00000	4.00000	.97727	.60537
0	0.00000	4.50000	1.14377	.62613
0	0.00000	5.00000	1.25948	.53351
0	0.00000	5.50000	1.25764	.41990
0	0.00000	6.00000	1.17726	.38706
0	0.00000	6.50000	1.12025	.45721
0	0.00000	7.00000	1.15781	.56034
0	0.00000	7.50000	1.27079	.60258
0	0.00000	8.00000	1.37321	.55043
0	0.00000	8.50000	1.39277	.45628
0	0.00000	9.00000	1.33412	.40665
0	0.00000	9.50000	1.27033	.44346
0	0.00000	10.00000	1.27423	.52919
0	0.00000	10.50000	1.35357	.58450
0	0.00000	11.00000	1.44747	.56120
0	0.00000	11.50000	1.48419	.48379
0	0.00000	12.00000	1.44590	.42428
0	0.00000	12.50000	1.38242	.43551
0	0.00000	13.00000	1.36367	.50452
0	0.00000	13.50000	1.41611	.56686
0	0.00000	14.00000	1.50061	.56649
0	0.00000	14.50000	1.55017	.50600
0	0.00000	15.00000	1.53129	.44212
0	0.00000	15.50000	1.47310	.43273
0	0.00000	16.00000	1.43894	.48457
0	0.00000	16.50000	1.46814	.54879
0	0.00000	17.00000	1.54120	.56687
0	0.00000	17.50000	1.59925	.52379
0	0.00000	18.00000	1.59847	.46033
0	0.00000	18.50000	1.54933	.43461
0	0.00000	19.00000	1.50562	.46894
0	0.00000	19.50000	1.51457	.53058
0	0.00000	20.00000	1.57421	.56289
0	0.00000	20.50000	1.63643	.53726
0	0.00000	21.00000	1.65180	.47836
0	0.00000	21.50000	1.61444	.44054
0	0.00000	22.00000	1.56634	.45764
0	0.00000	22.50000	1.55811	.51288
0	0.00000	23.00000	1.60286	.55519
0	0.00000	23.50000	1.66510	.54637
0	0.00000	24.00000	1.69407	.49553
0	0.00000	24.50000	1.67019	.44981
0	0.00000	25.00000	1.62226	.45071

n	COMPLEX ARGUMENT	S	INTEGRAL	$W_n(S)$
0	-.50000	0.00000	-.71150	.00000
0	-.50000	.50000	-.37536	.89327
0	-.50000	1.00000	.40747	1.30152
0	-.50000	1.50000	1.13222	1.08960
0	-.50000	2.00000	1.39273	.54823
0	-.50000	2.50000	1.16469	.14566
0	-.50000	3.00000	.77881	.16370
0	-.50000	3.50000	.62467	.51010
0	-.50000	4.00000	.83931	.84154
0	-.50000	4.50000	1.23010	.87298
0	-.50000	5.00000	1.47352	.60660
0	-.50000	5.50000	1.40207	.29969
0	-.50000	6.00000	1.13271	.22095
0	-.50000	6.50000	.94198	.41833
0	-.50000	7.00000	1.01396	.69521
0	-.50000	7.50000	1.28712	.79990
0	-.50000	8.00000	1.52554	.64979
0	-.50000	8.50000	1.53784	.39286
0	-.50000	9.00000	1.34347	.26299
0	-.50000	9.50000	1.14338	.36801
0	-.50000	10.00000	1.13194	.60094
0	-.50000	10.50000	1.32453	.74625
0	-.50000	11.00000	1.55280	.67724
0	-.50000	11.50000	1.62361	.46492
0	-.50000	12.00000	1.49289	.30564
0	-.50000	12.50000	1.29870	.33986
0	-.50000	13.00000	1.23001	.52859
0	-.50000	13.50000	1.35509	.69545
0	-.50000	14.00000	1.56484	.69045
0	-.50000	14.50000	1.67745	.52383
0	-.50000	15.00000	1.60469	.35124
0	-.50000	15.50000	1.42819	.32843
0	-.50000	16.00000	1.32026	.47089
0	-.50000	16.50000	1.38548	.64433
0	-.50000	17.00000	1.56846	.69061
0	-.50000	17.50000	1.70868	.57131
0	-.50000	18.00000	1.68872	.39897
0	-.50000	18.50000	1.53923	.33098
0	-.50000	19.00000	1.40711	.42595
0	-.50000	19.50000	1.41941	.59325
0	-.50000	20.00000	1.56877	.67908
0	-.50000	20.50000	1.72352	.60739
0	-.50000	21.00000	1.75020	.44693
0	-.50000	21.50000	1.63458	.34535
0	-.50000	22.00000	1.49180	.39339
0	-.50000	22.50000	1.45884	.54384
0	-.50000	23.00000	1.56972	.65756
0	-.50000	23.50000	1.72687	.63182
0	-.50000	24.00000	1.79264	.49286
0	-.50000	24.50000	1.71519	.36929
0	-.50000	25.00000	1.57389	.37310

n	COMPLEX ARGUMENT S		INTEGRAL $W_n(S)$	
0	-4.50000	0.00000	-1171.91416	.00000
0	-4.50000	.50000	-694.22874	937.68680
0	-4.50000	1.00000	338.46565	1100.70442
0	-4.50000	1.50000	1066.09227	368.35126
0	-4.50000	2.00000	898.23592	-631.76708
0	-4.50000	2.50000	5.84585	-1065.13447
0	-4.50000	3.00000	-841.09549	-597.31674
0	-4.50000	3.50000	-939.90104	339.26634
0	-4.50000	4.00000	-246.01031	936.65237
0	-4.50000	4.50000	614.22222	709.62027
0	-4.50000	5.00000	902.93642	-105.14773
0	-4.50000	5.50000	404.84083	-781.60721
0	-4.50000	6.00000	-411.25151	-747.46529
0	-4.50000	6.50000	-825.79649	-69.40497
0	-4.50000	7.00000	-500.71327	632.87822
0	-4.50000	7.50000	244.61147	748.01298
0	-4.50000	8.00000	739.78444	204.09806
0	-4.50000	8.50000	563.28526	-491.85207
0	-4.50000	9.00000	-99.28050	-721.99973
0	-4.50000	9.50000	-643.18231	-304.35061
0	-4.50000	10.00000	-593.48952	364.79418
0	-4.50000	10.50000	-18.73020	683.12419
0	-4.50000	11.00000	550.58425	382.52972
0	-4.50000	11.50000	609.06264	-246.00354
0	-4.50000	12.00000	123.90182	-630.95958
0	-4.50000	12.50000	-452.74516	-438.18609
0	-4.50000	13.00000	-603.02471	139.86004
0	-4.50000	13.50000	-207.25177	573.40440
0	-4.50000	14.00000	362.28904	479.33783
0	-4.50000	14.50000	589.79912	-41.02210
0	-4.50000	15.00000	282.01213	-507.64934
0	-4.50000	15.50000	-268.42425	-503.31927
0	-4.50000	16.00000	-559.72949	-45.92341
0	-4.50000	16.50000	-337.43287	440.10607
0	-4.50000	17.00000	183.93284	516.53711
0	-4.50000	17.50000	526.58051	125.77903
0	-4.50000	18.00000	386.47209	-367.46199
0	-4.50000	18.50000	-97.40479	-515.64973
0	-4.50000	19.00000	-479.58698	-193.85064
0	-4.50000	19.50000	-417.48619	295.63983
0	-4.50000	20.00000	21.98416	506.47919
0	-4.50000	20.50000	432.57978	254.69852
0	-4.50000	21.00000	443.62258	-221.22340
0	-4.50000	21.50000	54.21997	-485.54501
0	-4.50000	22.00000	-374.19081	-303.70069
0	-4.50000	22.50000	-452.76496	149.89134
0	-4.50000	23.00000	-117.78104	458.46660
0	-4.50000	23.50000	318.49587	345.38130
0	-4.50000	24.00000	458.42822	-78.18053
0	-4.50000	24.50000	181.06681	-421.80289
0	-4.50000	25.00000	-253.69624	-375.29716

n	COMPLEX ARGUMENT S		INTEGRAL $W_n(S)$	
1	0.00000	0.00000	0.00000	0.00000
1	0.00000	.50000	-.28433	.11740
1	0.00000	1.00000	-.39543	.38805
1	0.00000	1.50000	-.28715	.63003
1	0.00000	2.00000	-.06751	.69857
1	0.00000	2.50000	.09261	.58880
1	0.00000	3.00000	.09228	.42468
1	0.00000	3.50000	-.03169	.34996
1	0.00000	4.00000	-.15099	.41417
1	0.00000	4.50000	-.15994	.54517
1	0.00000	5.00000	-.05937	.62297
1	0.00000	5.50000	.05571	.58560
1	0.00000	6.00000	.08627	.47616
1	0.00000	6.50000	.01476	.39654
1	0.00000	7.00000	-.08622	.41446
1	0.00000	7.50000	-.12386	.50711
1	0.00000	8.00000	-.06772	.58745
1	0.00000	8.50000	.02766	.58493
1	0.00000	9.00000	.07615	.50668
1	0.00000	9.50000	.03805	.42669
1	0.00000	10.00000	-.04720	.41649
1	0.00000	10.50000	-.10022	.48171
1	0.00000	11.00000	-.07438	.56033
1	0.00000	11.50000	.00418	.58121
1	0.00000	12.00000	.06318	.52812
1	0.00000	12.50000	.05091	.45187
1	0.00000	13.00000	-.01825	.42200
1	0.00000	13.50000	-.07938	.46363
1	0.00000	14.00000	-.07727	.53658
1	0.00000	14.50000	-.01570	.57392
1	0.00000	15.00000	.04805	.54318
1	0.00000	15.50000	.05666	.47440
1	0.00000	16.00000	.00443	.43096
1	0.00000	16.50000	-.05920	.45136
1	0.00000	17.00000	-.07606	.51521
1	0.00000	17.50000	-.03199	.56342
1	0.00000	18.00000	.03159	.55278
1	0.00000	18.50000	.05676	.49457
1	0.00000	19.00000	.02196	.44283
1	0.00000	19.50000	-.03947	.44432
1	0.00000	20.00000	-.07092	.49632
1	0.00000	20.50000	-.04442	.55037
1	0.00000	21.00000	.01474	.55737
1	0.00000	21.50000	.05223	.51207
1	0.00000	22.00000	.03466	.45685
1	0.00000	22.50000	-.02060	.44209
1	0.00000	23.00000	-.06234	.48032
1	0.00000	23.50000	-.05278	.53562
1	0.00000	24.00000	-.00156	.55736
1	0.00000	24.50000	.04399	.52645
1	0.00000	25.00000	.04267	.47209

n	COMPLEX ARGUMENT	S	INTEGRAL	$W_n(S)$
1	-.50000	0.00000	-.00000	-.48815
1	-.50000	.50000	-.63143	-.21096
1	-.50000	1.00000	-.84297	.41838
1	-.50000	1.50000	-.53353	.95230
1	-.50000	2.00000	.01599	1.04552
1	-.50000	2.50000	.37932	.70916
1	-.50000	3.00000	.32163	.27254
1	-.50000	3.50000	-.03759	.09778
1	-.50000	4.00000	-.35575	.29116
1	-.50000	4.50000	-.36644	.64492
1	-.50000	5.00000	-.09008	.84026
1	-.50000	5.50000	.21106	.72436
1	-.50000	6.00000	.27673	.42474
1	-.50000	6.50000	.07115	.21694
1	-.50000	7.00000	-.20345	.27553
1	-.50000	7.50000	-.29934	.52999
1	-.50000	8.00000	-.14242	.74277
1	-.50000	8.50000	.11398	.72805
1	-.50000	9.00000	.23764	.51204
1	-.50000	9.50000	.12685	.29781
1	-.50000	10.00000	-.10659	.27605
1	-.50000	10.50000	-.24747	.45653
1	-.50000	11.00000	-.17355	.66825
1	-.50000	11.50000	.03989	.72017
1	-.50000	12.00000	.19601	.57262
1	-.50000	12.50000	.15757	.36626
1	-.50000	13.00000	-.03265	.28884
1	-.50000	13.50000	-.19730	.40500
1	-.50000	14.00000	-.18869	.60300
1	-.50000	14.50000	-.02037	.70149
1	-.50000	15.00000	.15075	.61505
1	-.50000	15.50000	.17048	.42782
1	-.50000	16.00000	.02620	.31208
1	-.50000	16.50000	-.14634	.37016
1	-.50000	17.00000	-.19001	.54428
1	-.50000	17.50000	-.06885	.67351
1	-.50000	18.00000	.10304	.64217
1	-.50000	18.50000	.16884	.48307
1	-.50000	19.00000	.07202	.34377
1	-.50000	19.50000	-.09519	.34997
1	-.50000	20.00000	-.17921	.49233
1	-.50000	20.50000	-.10573	.63831
1	-.50000	21.00000	.05493	.65538
1	-.50000	21.50000	.15504	.53111
1	-.50000	22.00000	.10529	.38158
1	-.50000	22.50000	-.04559	.34317
1	-.50000	23.00000	-.15815	.44829
1	-.50000	23.50000	-.13079	.59830
1	-.50000	24.00000	.00877	.65589
1	-.50000	24.50000	.13145	.57062
1	-.50000	25.00000	.12615	.42295

n	COMPLEX ARGUMENT	S	INTEGRAL	$W_n(S)$
2	0.00000	0.00000	0.00000	0.00000
2	0.00000	.50000	-.04791	.01981
2	0.00000	1.00000	-.13741	.13616
2	0.00000	1.50000	-.15372	.35472
2	0.00000	2.00000	-.02395	.57841
2	0.00000	2.50000	.21431	.68524
2	0.00000	3.00000	.43757	.62979
2	0.00000	3.50000	.53359	.48200
2	0.00000	4.00000	.48915	.37074
2	0.00000	4.50000	.39523	.38082
2	0.00000	5.00000	.36765	.49003
2	0.00000	5.50000	.45217	.59668
2	0.00000	6.00000	.59340	.61051
2	0.00000	6.50000	.69000	.52753
2	0.00000	7.00000	.68463	.42696
2	0.00000	7.50000	.61030	.39748
2	0.00000	8.00000	.55616	.46004
2	0.00000	8.50000	.58676	.55394
2	0.00000	9.00000	.68739	.59465
2	0.00000	9.50000	.78152	.54917
2	0.00000	10.00000	.80154	.46236
2	0.00000	10.50000	.74805	.41338
2	0.00000	11.00000	.68614	.44402
2	0.00000	11.50000	.68470	.52331
2	0.00000	12.00000	.75512	.57832
2	0.00000	12.50000	.84361	.56086
2	0.00000	13.00000	.88191	.48948
2	0.00000	13.50000	.84887	.43027
2	0.00000	14.00000	.78766	.43594
2	0.00000	14.50000	.76476	.49907
2	0.00000	15.00000	.80954	.56088
2	0.00000	15.50000	.88919	.56576
2	0.00000	16.00000	.94025	.51116
2	0.00000	16.50000	.92672	.44817
2	0.00000	17.00000	.87165	.43390
2	0.00000	17.50000	.83460	.47979
2	0.00000	18.00000	.85690	.54266
2	0.00000	18.50000	.92487	.56519
2	0.00000	19.00000	.98371	.52810
2	0.00000	19.50000	.98810	.46653
2	0.00000	20.00000	.94299	.43685
2	0.00000	20.50000	.89779	.46515
2	0.00000	21.00000	.90061	.52437
2	0.00000	21.50000	.95471	.56008
2	0.00000	22.00000	1.01665	.54044
2	0.00000	22.50000	1.03670	.48453
2	0.00000	23.00000	1.00411	.44390
2	0.00000	23.50000	.95596	.45510
2	0.00000	24.00000	.94256	.50686
2	0.00000	24.50000	.98144	.55131
2	0.00000	25.00000	1.04218	.54822

n	COMPLEX ARGUMENT S		INTEGRAL $W_n(S)$	
2	-.50000	0.00000	.08043	-.00000
2	-.50000	.50000	-.06682	-.14208
2	-.50000	1.00000	-.36419	-.01814
2	-.50000	1.50000	-.50451	.39824
2	-.50000	2.00000	-.28702	.86147
2	-.50000	2.50000	.20225	1.05722
2	-.50000	3.00000	.65386	.86699
2	-.50000	3.50000	.78766	.46831
2	-.50000	4.00000	.57899	.18689
2	-.50000	4.50000	.26858	.22941
2	-.50000	5.00000	.15316	.52400
2	-.50000	5.50000	.33982	.79826
2	-.50000	6.00000	.67138	.81948
2	-.50000	6.50000	.87757	.58735
2	-.50000	7.00000	.81257	.31839
2	-.50000	7.50000	.57215	.24649
2	-.50000	8.00000	.39646	.42014
2	-.50000	8.50000	.45351	.67123
2	-.50000	9.00000	.69696	.77366
2	-.50000	9.50000	.91827	.64426
2	-.50000	10.00000	.93828	.40855
2	-.50000	10.50000	.76374	.27992
2	-.50000	11.00000	.57272	.36693
2	-.50000	11.50000	.54929	.58183
2	-.50000	12.00000	.71984	.72690
2	-.50000	12.50000	.93620	.67491
2	-.50000	13.00000	1.01470	.47968
2	-.50000	13.50000	.90151	.32109
2	-.50000	14.00000	.71599	.33963
2	-.50000	14.50000	.63771	.51198
2	-.50000	15.00000	.74344	.67763
2	-.50000	15.50000	.94171	.68745
2	-.50000	16.00000	1.06047	.53727
2	-.50000	16.50000	1.00431	.36710
2	-.50000	17.00000	.83800	.33071
2	-.50000	17.50000	.72343	.45678
2	-.50000	18.00000	.77095	.62660
2	-.50000	18.50000	.94125	.68530
2	-.50000	19.00000	1.08499	.58259
2	-.50000	19.50000	1.08054	.41538
2	-.50000	20.00000	.94321	.33645
2	-.50000	20.50000	.80795	.41487
2	-.50000	21.00000	.80437	.57561
2	-.50000	21.50000	.93954	.67090
2	-.50000	22.00000	1.09452	.61570
2	-.50000	22.50000	1.13496	.46334
2	-.50000	23.00000	1.03321	.35403
2	-.50000	23.50000	.89105	.38591
2	-.50000	24.00000	.84461	.52690
2	-.50000	24.50000	.94016	.64658
2	-.50000	25.00000	1.09400	.63655

n	COMPLEX ARGUMENT	S	INTEGRAL	$W_n(S)$
10	-.50000	0.00000	.00160	-.00000
10	-.50000	.50000	.00004	-.00321
10	-.50000	1.00000	-.00475	-.00665
10	-.50000	1.50000	-.01307	-.01058
10	-.50000	2.00000	-.02555	-.01553
10	-.50000	2.50000	-.04378	-.02249
10	-.50000	3.00000	-.07173	-.03239
10	-.50000	3.50000	-.11718	-.04328
10	-.50000	4.00000	-.18998	-.04497
10	-.50000	4.50000	-.29373	-.01498
10	-.50000	5.00000	-.41267	.07663
10	-.50000	5.50000	-.50372	.24901
10	-.50000	6.00000	-.50705	.48566
10	-.50000	6.50000	-.37880	.72211
10	-.50000	7.00000	-.13062	.86523
10	-.50000	7.50000	.15488	.84420
10	-.50000	8.00000	.35574	.66465
10	-.50000	8.50000	.38109	.42382
10	-.50000	9.00000	.23661	.26320
10	-.50000	9.50000	.03348	.28011
10	-.50000	10.00000	-.07747	.45754
10	-.50000	10.50000	-.00791	.66891
10	-.50000	11.00000	.20250	.76341
10	-.50000	11.50000	.41142	.67517
10	-.50000	12.00000	.47765	.47373
10	-.50000	12.50000	.36793	.31317
10	-.50000	13.00000	.18316	.31442
10	-.50000	13.50000	.08030	.47124
10	-.50000	14.00000	.14882	.65543
10	-.50000	14.50000	.34004	.71934
10	-.50000	15.00000	.50730	.61456
10	-.50000	15.50000	.52567	.43011
10	-.50000	16.00000	.39368	.31829
10	-.50000	16.50000	.23303	.36878
10	-.50000	17.00000	.18636	.53527
10	-.50000	17.50000	.29832	.67589
10	-.50000	18.00000	.47968	.67330
10	-.50000	18.50000	.58435	.53288
10	-.50000	19.00000	.53368	.37635
10	-.50000	19.50000	.38206	.33668
10	-.50000	20.00000	.26853	.44476
10	-.50000	20.50000	.29664	.60463
10	-.50000	21.00000	.44634	.67782
10	-.50000	21.50000	.59317	.60304
10	-.50000	22.00000	.61733	.44779
10	-.50000	22.50000	.50717	.34790
10	-.50000	23.00000	.36751	.38873
10	-.50000	23.50000	.32677	.53178
10	-.50000	24.00000	.42473	.65024
10	-.50000	24.50000	.57929	.64103
10	-.50000	25.00000	.66016	.51450

n	COMPLEX ARGUMENT	S	INTEGRAL	$W_n(S)$
10	0.00000	0.00000	0.00000	0.00000
10	0.00000	.50000	-.00162	.00000
10	0.00000	1.00000	-.00658	.00000
10	0.00000	1.50000	-.01525	.00000
10	0.00000	2.00000	-.02840	.00004
10	0.00000	2.50000	-.04752	.00037
10	0.00000	3.00000	-.07514	.00219
10	0.00000	3.50000	-.11468	.00916
10	0.00000	4.00000	-.16838	.02918
10	0.00000	4.50000	-.23305	.07445
10	0.00000	5.00000	-.29530	.15715
10	0.00000	5.50000	-.33076	.28039
10	0.00000	6.00000	-.31177	.42872
10	0.00000	6.50000	-.22311	.56618
10	0.00000	7.00000	-.07759	.64851
10	0.00000	7.50000	.08113	.64662
10	0.00000	8.00000	.19586	.56838
10	0.00000	8.50000	.22746	.46136
10	0.00000	9.00000	.18110	.38953
10	0.00000	9.50000	.10703	.39484
10	0.00000	10.00000	.07058	.46918
10	0.00000	10.50000	.10907	.55861
10	0.00000	11.00000	.20591	.60009
10	0.00000	11.50000	.30241	.56637
10	0.00000	12.00000	.34082	.48575
10	0.00000	12.50000	.30747	.42060
10	0.00000	13.00000	.24268	.41979
10	0.00000	13.50000	.20916	.48126
10	0.00000	14.00000	.24286	.55446
10	0.00000	14.50000	.32509	.58100
10	0.00000	15.00000	.39850	.54136
10	0.00000	15.50000	.41398	.46989
10	0.00000	16.00000	.37042	.42584
10	0.00000	16.50000	.31472	.44445
10	0.00000	17.00000	.30218	.50835
10	0.00000	17.50000	.35058	.56306
10	0.00000	18.00000	.42606	.56313
10	0.00000	18.50000	.47263	.51009
10	0.00000	19.00000	.45976	.45018
10	0.00000	19.50000	.40773	.43437
10	0.00000	20.00000	.36939	.47494
10	0.00000	20.50000	.38449	.53579
10	0.00000	21.00000	.44592	.56425
10	0.00000	21.50000	.50673	.53666
10	0.00000	22.00000	.52135	.47815
10	0.00000	22.50000	.48489	.44002
10	0.00000	23.00000	.43665	.45486
10	0.00000	23.50000	.42520	.50860
10	0.00000	24.00000	.46595	.55359
10	0.00000	24.50000	.52835	.55076
10	0.00000	25.00000	.56339	.50358

APPENDIX B

Fortran Program Listing Used to Evaluate $W_n(S)$

```
      PROGRAM TIMED(INPUT,OUTPUT)
      COMPLEX SM,SNZ,Z,S,AJ
      AJ=CMPLX(0.0,1.0)
      DO 300 NNX=8,11
      N=NNX-1
      DO 200 IMA=1,11
      SR=0.5*FLOAT(IMA-1)
      SI=0.5
      PRINT 151
  151 FORMAT(*1*,//)
      PRINT 150
  150 FORMAT(18X,*N*,6X,*COMPLEX ARGUMENT  S*,8X,*INTEGRAL  SN(N,S)*,//)
      DO 100 I=1,51
      S=-SR+AJ*SI*FLOAT(I-1)
      SNZ=SM(N,S)
      PRINT 130, N,S,SNZ
  130 FORMAT(15X,I14,4F13.5)
  100 CONTINUE
  200 CONTINUE
  300 CONTINUE
      STOP 10
      END
      COMPLEX FUNCTION SM(M,S)
C     SM(M,S) COMPUTES THE INTEGRAL OF THE ANGER-WEBER FUNCTION BY
C     SERIES FOR COMPLEX ARGUMENT)
      COMPLEX J,X,S
      DOUBLE C,A1,ZR,ZI,SR1,SI1,SR2,SI2,ZR2,ZI2,ZRL,ZIL,TR1,TI1,TR2,TI2,
     $TRP,TIP,ZMAG,ZANG,Y1,Y2,PML(350),A2,A(250),B(250),SQPI,AT1,AT2,AS1
     $,AS2
      DATA MSAVE/-1/,NUM/188/,J/(0.,1.)/
      DATA IFLAG/-1/,SQPI/1.7724538509055160272981674833D0/
      X=-J*S
      XR=REAL(X)
      XI=AIMAG(X)
      IF(IFLAG)100,500,500
100   A(62)=SQPI
      DO 200 I=1,61
200   A(62-I)=DBLE(-2./FLOAT(2*I-1))*A(63-I)
      DO 300 I=1,175
300   A(62+I)=DBLE(.5*FLOAT(2*I-1))*A(61+I)
      B(1)=1.D0
      B(2)=B(1)
      DO 400 I=2,175
400   B(I+1)=DBLE(FLOAT(I))*B(I)
      IFLAG=1
500   IF(MSAVE-M)1,9,1
1     C=0.0D0
      DO 51 IL=1,NUM
      L=IL-1
      IF(MOD(L+M,2))2,3,2
2     A1=A((L+M+125)/2)
      A2=A((L-M+125)/2)
      GO TO 42
3     NAG=(L-M+2)/2
      IF(NAG)4,4,41
4     PML(IL)=0.0D0
```

```
      GO TO 51
41    A1=B(NAG)
      A2=B((L+M+2)/2)
42    IF(DABS(C).GT.1.D300)GO TO 4
      C=DBLE(FLOAT(IL))*A1*A2
      PML(IL)=1.D0/C
51    CONTINUE
      MSAVE=M
      NU=NUM-3
9     CONTINUE
      IF(CABS(X).GE.35.) GO TO 117
      SR1=0.0D0
      SI1=0.0D0
      SR2=0.0D0
      SI2=0.0D0
      ZR=DBLE(XR)
      ZI=DBLE(XI)
      ZR2=ZR*ZR-ZI*ZI
      ZI2=2.D0*ZR*ZI
      DO 10 IL=1,NU,4
      N=IL+3
      ZMAG=DSQRT(ZR*ZR+ZI*ZI)
      IF(ZR.NE.0.0D0) GO TO 30
      ZANG=SQPI*SQPI*0.5D0
      GO TO 31
30    ZANG=DATAN2(ZI,ZR)
31    Y1=ZMAG**IL
      Y2=ZANG*DBLE(FLOAT(IL))
      ZRL=Y1*DCOS(Y2)
      ZIL=Y1*DSIN(Y2)
      TR1=PML(IL)*ZRL+PML(IL+2)*ZI2*ZIL-PML(IL+2)*ZR2*ZRL
      TI1=PML(IL)*ZIL-PML(IL+2)*ZR2*ZIL-PML(IL+2)*ZI2*ZRL
      TRP=PML(IL+1)*ZRL+PML(IL+3)*ZI2*ZIL-PML(IL+3)*ZR2*ZRL
      TIP=PML(IL+1)*ZIL-PML(IL+3)*ZR2*ZIL-PML(IL+3)*ZI2*ZRL
      TR2=TRP*ZR-TIP*ZI
      TI2=TIP*ZR+ZI*TRP
      SR1=SR1+TR1
      SI1=SI1+TI1
      SR2=SR2+TR2
      SI2=SI2+TI2
      AT1=DSQRT(TR1*TR1+TI1*TI1)
      AT2=DSQRT(TR2*TR2+TI2*TI2)
      AS1=DSQRT(SR1*SR1+SI1*SI1)
      AS2=DSQRT(SR2*SR2+SI2*SI2)
      IF(AS1.EQ.0.0D0) GO TO 10
      IF(AS2.EQ.0.0D0) GO TO 10
      IF(DABS(AT1/AS1).GE.1.D-7) GO TO 10
      IF(DABS(AT2/AS2).GE.1.D-7) GO TO 10
      GO TO 11
10    CONTINUE
11    S4=SR2-SI1
      S3=SI2+SR1
      SM=CMPLX(S4,S3)*J**M
      RETURN
117   SM=(0.,0.)
      RETURN
      END
```