Mathematics Notes

Note 43

December 1975

Quaternion Calculus and the Solution of Maxwell's Equations

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ABSTRACT

We present a review of the essential features of quanternion algebra and calculus. These are then applied to electromagnetic theory and some of the interesting features of this application are discussed. Examples are presented to illustrate how this formalism yields some of the more fundamental solutions of Maxwell's equations; i.e., static multipole fields and time harmonic point dipole fields. Areas of possible future development are discussed.

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Introduction

When James Clerk Maxwell formalized electromagnetics in his celebrated equations[1] he wrote them, among other ways, in terms of Hamilton's quaternions.[2-5] However, as is so often the case, he was apparently unaware of the full implication of what he had done. It remained a curiosity unitl the 1930's when Rudolf Fueter and his students and colleagues[6-13] developed a calculus of quaternions much akin to complex variable theory even to the extent of a four dimensional analog of the Cauchy-Riemann equations. Thereafter, it has been noticed now and again by various investigators that, with proper identification of variables, these "Cauchy-Riemann like" equations (the so-called regularity conditions) are in fact nothing but Maxwell's equations in free space. However, it seems that this fact has never been exploited to any great extent. It is, of course, not very difficult to find possible reasons for this; not the least of which is the vast generality and universal acceptance of the tensor formalism which has all but obliterated the remarkably ingenious work of Hamilton.[14]

The purpose of this exposition is primarily to demonstrate techniques for manipulating quaternions in electromagnetic theory. Quaternion calculus is an area somewhat alien to many people and it is not always clear just what must be done to obtain concrete results. One may, for example, speak of the abstract properties of vectors and of the beauty of the vector calculus but when it comes to teaching, sooner or later the "nitty gritty" manipulation of components must be presented. Similarly, one may speak in generalities and formulate equations and thereby demonstrate the elegance and mystique of these quaternions, but sooner or later, one must understand the details of manipulating them to obtain answers.

We present here the results of our first efforts to make use of the quaternion formalism in classical electromagnetic theory. We begin by summarizing for reference the previously existing real quaternion formalism. In adapting this existing formalism to the equations of electromagnetics, one must make a generalization from real quaternions to complex quaternions. Not all of the properties of real quaternions carry over to complex quaternions, but many do, and it is these which provide the basis for the calculational techniques described subsequently. We ask then that the reader bear with us as we first present the theory of real quaternions even though it is complex quaternions which we will later require. Finally, we present here several of our results indicating what can be done at this point in the development of the theory.

Quaternion Algebra

A quaternion is, in essence, a four component "complex number." That is it has a real part and three distinct imaginary parts and thus it may be written

$$\mathbf{a} = a_0 + a_1\hat{i}_1 + a_2\hat{i}_2 + a_3\hat{i}_3 = a_0 + \vec{a} \qquad (1)$$

where a_0, a_1, a_2, a_3 are real numbers and $\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3$ are unit imaginaries, i.e., $\hat{i}_1^{\,2} = \hat{i}_2^{\,2} = \hat{i}_3^{\,2} = -1$. a_0 is termed the scalar part and $\vec{a}$ the vector part of the quaternion. A quaternion may be multiplied by a scalar and the product is commutative and distributive; thus,

$$b_0\mathbf{a} = \mathbf{a}b_0 = b_0a_0 + b_0\vec{a} = b_0a_0 + b_0a_1\hat{i}_1 + b_0a_2\hat{i}_2 + b_0a_3\hat{i}_3 \qquad (2)$$

where b_0 is any scalar (real or complex). Quaternions are added by addition of their corresponding components so that

$$\mathbf{a} + \mathbf{b} = (a_0 + b_0) + (a_1 + b_1)\hat{i}_1 + (a_2 + b_2)\hat{i}_2 + (a_3 + b_3)\hat{i}_3 \qquad (3)$$

The algebra of these quaternions can be described by means of the following relation.

$$\hat{i}_1\hat{i}_2\hat{i}_3 = -1 \qquad (4)$$

From this may be derived the rules for multiplying quaternions. For example, in multiplying two quaternions we might require the product $\hat{i}_2\hat{i}_3$. We obtain this by multiplying equation (4) by $\hat{i}_1$ which yields

$$\hat{i}_1\hat{i}_1\hat{i}_2\hat{i}_3 = -\hat{i}_1 \qquad (5)$$

Now $\hat{i}_1\hat{i}_1 = \hat{i}_1^{\,2} = -1$ so that

$$\hat{i}_2\hat{i}_3 = \hat{i}_1 \qquad (6)$$

Note that using a similar procedure (multiplying from the right first by $\hat{i}_3$ and then by $\hat{i}_2$) one may easily show that $\hat{i}_3\hat{i}_2 = -\hat{i}_1$ which clearly demonstrates that the algebra of quaternions is a noncommutative one. We are now in a position to write down the product of two quaternions.

$$
\begin{aligned}
\mathbf{ab} &= (a_0+a_1\hat{i}_1+a_2\hat{i}_2+a_3\hat{i}_3)(b_0+b_1\hat{i}_1+b_2\hat{i}_2+b_3\hat{i}_3) \\
&= (a_0b_0-a_1b_1-a_2b_2-a_3b_3) \\
&\quad + (a_0b_1+a_1b_0+a_2b_3-a_3b_2)\hat{i}_1 \\
&\quad + (a_0b_2+a_2b_0+a_3b_1-a_1b_3)\hat{i}_2 \\
&\quad + (a_0b_3+a_3b_0+a_1b_2-a_2b_1)\hat{i}_3 \\
&= a_0b_0-\vec{a}\cdot\vec{b}+a_0\vec{b}+b_0\vec{a}+\vec{a}\times\vec{b}
\end{aligned}
\tag{7}
$$

where the dot and cross products indicated are those of ordinary vector analysis. Note that this again indicates the noncommutativity of the quaternion product.

The conjugate of a quaternion is defined in analogy with complex variable theory; that is,

$$\mathbf{a}^* = a_0 - \vec{a} \tag{8}$$

and the squared magnitude or norm of a quaternion follows naturally as

$$|\mathbf{a}|^2 = \mathbf{aa}^* = a_0^2+a_1^2+a_2^2+a_3^2 \tag{9}$$

and is obviously positive (and zero only if $\mathbf{a} = 0$). Thus we can conclude that

$$a_0 = \tfrac{1}{2}(\mathbf{a}+\mathbf{a}^*)$$

$$\vec{a} = \tfrac{1}{2}(\mathbf{a}-\mathbf{a}^*)$$

$$(\mathbf{ab})^* = \mathbf{b}^*\mathbf{a}^* \tag{10}$$

The inverse of a quaternion is defined to be such that

$$\mathbf{aa}^{-1} = 1 \tag{11}$$

Thus,

$$\mathbf{a}^*\mathbf{aa}^{-1} = \mathbf{a}^* \quad \text{or}$$

$$\mathbf{a}^{-1} = \frac{\mathbf{a}^*}{|\mathbf{a}|^2} \tag{12}$$

Hence, every nonzero quaternion has a unique inverse and we note that

$$(\mathbf{ab})^{-1} = \mathbf{b}^{-1}\mathbf{a}^{-1} \tag{13}$$

The quaternions, then, form a division algebra which is noncommutative but for which all the other properties of the algebra of real numbers and complex numbers hold. It has been shown that there is only one other division algebra, that of octonions (eight component numbers) but that algebra is both noncommutative and nonassociative. (This is not a bad as it sounds; the vector product for example is neither commutative nor associative.) Were we to allow the components of the quaternions to be complex numbers we would no longer retain the division algebra. That is there exist non-zero complex quaternions which have no inverse because their norms are zero (c.f., there exist matrices which have no inverses because their determinants are zero).

Quaternion Calculus

At this point we introduce the concept of a function of a quaternion variable $\mathbf{x}$, i.e. $\mathbf{u}(\mathbf{x})$. One might now proceed as in complex variable theory and require that the change in $\mathbf{u}$ for a given change in $\mathbf{x}$ be independent of the direction of the change in $\mathbf{x}$; in other words one might try to invent a unique derivative. However, if this is done the requirements on $\mathbf{u}$ are so stringent that only a few functions (such as constants) can satisfy them. We proceed, instead, in analogy with Morera's theorem of complex variable theory. That is, we call a function "regular" in a four-volume V if

$$\oint_H \mathbf{u}\, d\mathbf{q} = 0 \tag{14}$$

for every closed hypersurface H in V†. Note that this is not the same as requiring that

$$\oint_H (d\mathbf{q})\mathbf{u} = 0 \tag{15}$$

for every closed H in V because the algebra is not commutative. Thus, $\mathbf{u}$'s satisfying (14) are termed "right regular" and $\mathbf{u}$'s satisfying (15) are termed "left regular."

There is, in this calculus an analog of the Gauss theorem of vector calculus[7] and introducing the differential operator $\mathrm{II} = \frac{\partial}{\partial x_0} - \nabla = \frac{\partial}{\partial x_0} - \hat{i}_1 \frac{\partial}{\partial x_1} - \hat{i}_2 \frac{\partial}{\partial x_2} - \hat{i}_3 \frac{\partial}{\partial x_3}$ and its conjugate $\mathrm{II}^* = \frac{\partial}{\partial x_0} + \nabla$, we may write this analog as

$$\oint_H (d\mathbf{q})\mathbf{u} = \int_V \mathrm{II}^* \mathbf{u}\, dv \tag{16}$$

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†$d\mathbf{q}$ is the outwardly directed element of the three-surface H bounding the four volume V.

One may readily demonstrate this equality by means of the Gauss theorem as follows. First we write the elemental three-surface element $d\mathbf{q}$ in detail as

$$d\mathbf{q} = \sum_{k=0}^{3} \xi_k \hat{i}_k \quad dh$$

where dh is the "3-area" of the element and the quaternion in parenthesis is a unit quaternion in the direction of the outward normal to the closed three surface H. Now, making use of (7) we form $(d\mathbf{q})\mathbf{u}$ and integrate it over H as required on the left side of (16).

$$\oint_H (d\mathbf{q})\mathbf{u} = \oint_H (\xi_o u_o - \xi_1 u_1 - \xi_2 u_2 - \xi_3 u_3) dh$$

$$+ \hat{i}_1 \oint_H (\xi_o u_1 + \xi_1 u_o + \xi_2 u_3 - \xi_3 u_2) dh$$

$$+ \hat{i}_2 \oint_H (\xi_o u_2 + \xi_2 u_6 + \xi_3 u_1 - \xi_1 u_3) dh$$

$$+ \hat{i}_3 \oint_H (\xi_o u_3 + \xi_3 u_o + \xi_1 u_2 - \xi_2 u_1) dh$$

Applying the conventional four dimensional Gauss theorem to each of the above integrals we obtain;

$$\oint_H (d\mathbf{q})\mathbf{u} = \int_V \frac{\partial u_o}{\partial x_o} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \quad dv$$

$$+ \hat{i}_1 \int_V \frac{\partial u_1}{\partial x_o} + \frac{\partial u_o}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \quad dv$$

$$+ \hat{i}_2 \int_V \frac{\partial u_2}{\partial x_o} + \frac{\partial u_o}{\partial x_2} + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \quad dv$$

$$+ \hat{i}_3 \int_V \frac{\partial u_3}{\partial x_o} + \frac{\partial u_o}{\partial x_3} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad dv$$

$$= \int_V II^* \mathbf{u} dv$$

where dv is the scalar element of four dimensional volume in the region V. Analogously, one may show that

$$\oint_H \mathbf{u}\,d\mathbf{q} = \int_V \mathbf{u}\,\Pi^* dv \qquad (17)$$

Thus, a function $\mathbf{u}$ is right regular if

$$\mathbf{u}\Pi^* = 0 \qquad (18)$$

and left regular if

$$\Pi^*\mathbf{u} = 0 \qquad (19)$$

These regularity conditions may also be written in terms of the familiar vector differential operations. For right regularity,

$$\frac{\partial u_0}{\partial x_0} - \nabla\cdot\vec{u} + \frac{\partial\vec{u}}{\partial x_0} + \nabla u_0 - \nabla x\vec{u} = 0 \qquad (20)$$

and for left regularity,

$$\frac{\partial u_0}{\partial x_0} - \nabla\cdot\vec{u} + \frac{\partial\vec{u}}{\partial x_0} + \nabla u_0 + \nabla x\vec{u} = 0 \qquad (21)$$

That the regularity conditions are analogous to the Cauchy-Riemann equations may be made even more obvious by writing them out in detail as follows. For right regularity,

$$\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} = 0$$

$$\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} = 0$$

$$\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} = 0$$

$$\frac{\partial u_3}{\partial x_0} + \frac{\partial u_0}{\partial x_3} + \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0 \tag{22}$$

and for left regularity,

$$\begin{gathered}
\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} = 0 \\
\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0 \\
\frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = 0 \\
\frac{\partial u_3}{\partial x_0} + \frac{\partial u_0}{\partial x_3} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0
\end{gathered} \tag{23}$$

Note now that if in either (22) or (23) one sets any two of u_1, u_2, and u_3 equal to zero and asserts that u_0 and the remaining component of $\mathbf{u}$ are independent of the components of $\mathbf{x}$ corresponding to the zero components of $\mathbf{u}$, one obtains the Cauchy-Riemann equations. For example, if we set u_1 and u_2 equal to zero and assert that u_0 and u_3 are independent of x_1 and x_2, both (22) and (23) reduce to

$$\begin{gathered}
\frac{\partial u_0}{\partial x_0} = \frac{\partial u_3}{\partial x_3} \\
\frac{\partial u_3}{\partial x_0} = - \frac{\partial u_0}{\partial x_3}
\end{gathered} \tag{24}$$

Maxwell's Equations

If one now attempts to apply the above formalism to the variables of electromagnetics, one soon discovers a fundamental discrepancy between the laws governing the electromagnetic interaction and those governing the behavior of functions of a quaternion variable. This is due to the fact that, while x_0, x_1, x_2, x_3 are essentially alike in the world of real quaternions, the time variable is a fundamentally different sort of object than the three spatial variables in the world of four dimensional electrodynamics. We remedy this difficulty in the same way that Minkowski did in his approach to special relativity theory. That is, we introduce an imaginary variable, i t, where t is real, and substitute it for one of our four real variables, x_0, in the theory of real quaternions. This makes the replaced variable fundamentally different from the others in just the proper way, but it makes our quaternions complex. These complex quaternions then have in reality eight components but, because the components are paired into four complex components, these objects are not the nonassociative octonions. Rather, they are a special subcatagory of the octonions which retains the associative property. They no longer form a division algebra, however.

We shall find it convenient to work in units where the speed of light in vacuum, c, is unity; that is, we measure in time in meters (1 second = 3×10^8 meters).

Now, if $\mathbf{u}$ is left regular, then $\mathbf{v} = \Pi\mathbf{u}$ is also left regular because $\Pi^*\mathbf{v} = \Pi^*\Pi\mathbf{u} = \Pi(\Pi^*\mathbf{u}) = 0$. Let us make the substitution $x_0 = i\,t$ where $i = \sqrt{-1}$ and write out this equation in detail:††

$$(\frac{1}{i}\frac{\partial}{\partial t} - \nabla)(\frac{1}{i}\frac{\partial}{\partial t} + \nabla)\mathbf{u} = 0 \tag{25}$$

or

$$\Box\Box^*\mathbf{u} = 0 \quad \text{where } \Box = \frac{1}{i}\frac{\partial}{\partial t} - \nabla.$$

Now, introducing $i\,t$ makes $\mathbf{u}$ a complex quaternion (no division algebra) and we may write

$$\mathbf{u} = \boldsymbol{\phi} + i\mathbf{A} \tag{26}$$

where $\boldsymbol{\phi}$ and $\mathbf{A}$ are real quaternions and (25) becomes

$$(\nabla^2 - \frac{\partial^2}{\partial t^2})(\boldsymbol{\phi}+i\mathbf{A}) = 0 \tag{27}$$

Thus if $\mathbf{u} = \boldsymbol{\phi} + i\mathbf{A}$ is left regular then it satisfies the three dimensional wave equation. Writing out this left regularity requirement in detail we obtain,

$$\begin{aligned}
\Box^*(\boldsymbol{\phi}+i\mathbf{A}) &= (\frac{\partial\mathbf{A}}{\partial t} + \nabla\boldsymbol{\phi}) + i(\nabla\mathbf{A} - \frac{\partial\boldsymbol{\phi}}{\partial t}) \\
&= (\frac{\partial A_0}{\partial t} - \nabla\cdot\vec{\phi}) \\
&\quad + (\frac{\partial\vec{A}}{\partial t} + \nabla\phi_0 + \nabla\times\vec{\phi}) \\
&\quad + i(-\nabla\cdot\vec{A} - \frac{\partial\phi_0}{\partial t}) \\
&\quad + i(\nabla A_0 + \nabla\times\vec{A} - \frac{\partial\vec{\phi}}{\partial t}) = 0
\end{aligned} \tag{28}$$

††Alternatively, one may introduce the Minkowski metric at the outset by modifying the quaternion algebra asserting that $\hat{i}_1\hat{i}_2\hat{i}_3 = +i$ while $\hat{i}_k^2 = +1$.[15]

The four terms in parentheses represent respectively the real scalar, the real vector, the imaginary scalar, and the imaginary vector parts of the expression. In order for the expression to be zero each of these four quantities must individually be zero. We can satisfy this requirement by defining

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi_0 = \nabla x\vec{\phi} \tag{29}$$

$$\vec{B} = \nabla x\vec{A} = \frac{\partial \vec{\phi}}{\partial t} - \nabla A_0 \tag{30}$$

and requiring that

$$\frac{\partial \phi_0}{\partial t} + \nabla\cdot\vec{A} = 0 \tag{31}$$

and

$$\frac{\partial A_0}{\partial t} - \nabla\cdot\vec{\phi} = 0 \tag{32}$$

Thus ϕ_0 and $\vec{A}$ are the familiar scalar and vector potential satisfying the Lorentz gauge condition (31) while $\vec{\phi}$ and A_0 are the corresponding vector and scalar antipotentials satisfying the Lorentz gauge condition (32). We see, then, that the left regularity of $\mathbf{u} = \boldsymbol{\phi}+i\mathbf{A}$ is equivalent to Maxwell's equations because all components of $\boldsymbol{\phi}$ and $\mathbf{A}$ satisfy the wave equation (27) and the Lorentz gauge conditions (31) and (32).

The above result can be obtained more directly by introducing the so-called field quaternion

$$\mathbf{F} = \tfrac{1}{2}(\Box^*\mathbf{u}-\mathbf{u}\Box^*) = -\tfrac{1}{2}\mathbf{u}\Box^* = \vec{E} + i\vec{B} \tag{33}$$

Now,

$$\Box^*\mathbf{F} = -\tfrac{1}{2}\Box^*\mathbf{u}\Box^* = 0 \tag{34}$$

and,

$$\Box^*(\vec{E}+i\vec{B}) = (-\nabla\cdot\vec{E}) + (\frac{\partial\vec{B}}{\partial t} + \nabla x\vec{E})$$

$$+ i(-\nabla\cdot\vec{B}) + i(\nabla x\vec{B} - \frac{\partial\vec{E}}{\partial t}) = 0 \qquad (35)$$

which implies,

$$\nabla\cdot\vec{E} = 0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla x\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla x\vec{B} = \frac{\partial\vec{E}}{\partial t} \qquad (36)$$

which are, of course, Maxwell's equations in vacuum. Note that if **u** were both left and right (fully) regular, **F** would be identically zero, i.e. no fields can be obtained from a fully regular quaternion potential. Note also that it is easy to see from the left regularity of **F** (Equation 34) that

$$\Box\Box^*\mathbf{F} = \Box^2\mathbf{F} = 0 \qquad (37)$$

which implies that,

$$\Box^2\vec{E} = 0 \quad \text{and} \quad \Box^2\vec{B} = 0 \qquad (38)$$

That is, both $\vec{E}$ and $\vec{B}$ satisfy the vector wave equation.

We comment at this point that if by any means whatever we find a left regular complex quaternion **u** = **ϕ+iA** it will be a valid potential for **F**. However, it is not obvious that the substitution of $-ix_0$ for t will make **u** real. Thus, it may be that in order to cover all possible potentials we must allow **u** to be complex before substituting i t for x_0. This certainly makes the theory more flexible but perhaps it makes it too flexible (not to mention the demise of the division algebra). That is, on the basis of simplicity, one might argue as follows. Were it a fact that all physical potentials can be derived from real quaternions, then in static problems (t and x_0 absent) **A** would be identically zero. This would preclude the existence of a magnetic field in the absence of time dependence and hence would preclude the existence of magnetic poles. What we are saying is that if a magnetic pole were to be discovered, we would immediately know that we must allow $\mathfrak{u}$ to be complex before the substitution of $i\,t$ for x_0. There is no way to argue against the existence of magnetic poles on the basis of Maxwell's equations but the simplest quaternion formalism consistent with Maxwell's equations appears to preclude the existence of such poles.

A Simple Example

We have seen that electromagnetic fields satisfying Maxwell's equations can be derived from quaternion potentials. These potentials are rather alien to our usual way of thinking. Thus it may be well to present an example of a quaternion potential for a familiar field. The potential,

$$\mathfrak{u} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{\cot\theta}{r}\,\hat{i}_\phi\right] \tag{39}$$

yields the field of a point charge q. Its scalar part is the familiar Coulomb scalar potential while its vector part is the corresponding vector antipotential. $\hat{i}_\phi$ is a quaternion unit vector in the ϕ direction in a spherical coordinate system and θ is the polar angle in that system.

Potentials for More Complicated Static Fields

Since regular quaternion functions represent electromagnetic fields, we are now faced with the task of generating such functions. Two examples of quaternion potentials somewhat more substantial than the above example have been set down by Alan Rose[16] in a calculation of fluid flow around a hard sphere. They are

$$\mathbf{u} = -x_3 - \frac{x_2}{2}\hat{i} + \frac{x_1}{2}\hat{i}_2 = \boldsymbol{\beta}^{(1)} \tag{40}$$

and,

$$\mathbf{u} = \frac{-x_3+x_2\hat{i}_1-x_1\hat{i}_2}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \boldsymbol{\beta}^{(-1)} \tag{41}$$

The first, (40), yields a uniform $\hat{i}_3$ directed electric field while the second, (41), yields the field of an $\hat{i}_3$ directed electrostatic dipole. Note first that these potentials are time independent so the $x_0 \rightarrow i\,t$ substitution has no relevance here. Note also that in each case the $\hat{i}_3$ component is zero. This selects a preferred direction in space for these potentials which might just as well have been that of $\hat{i}_1$ or $\hat{i}_2$ or any other direction. However, we could not have set the scalar component equal to zero for then regularity would require that both the divergence and curl of the vector part be zero yielding no fields.

Consider Equation (40). This potential is a linear form in x_1, x_2, and x_3 having no $\hat{i}_3$ component. If we define a quaternion $\mathbf{z} = \hat{i}_z\vec{r}$ where $\vec{r} = \hat{i}_x x + \hat{i}_y y + \hat{i}_z z$, that is, $\mathbf{z} = -z + \hat{i}_y x - \hat{i}_x y$, then we may write $\boldsymbol{\beta}^{(1)}$ as

$$\boldsymbol{\beta}^{(1)} = \frac{3}{4}\mathbf{z} + \frac{1}{4}\mathbf{z}^* \tag{42}$$

which is a linear form in $\mathbf{z}$ and $\mathbf{z}^*$. We may now extend this scheme by postulating the following quadratic form in $\mathbf{z}$ and $\mathbf{z}^*$.

$$\boldsymbol{\beta}^{(2)} = A\mathbf{z}^2 + B\mathbf{z}\mathbf{z}^* + C\mathbf{z}^{*2} \tag{43}$$

Note that $\mathbf{z}$ commutes with $\mathbf{z}^*$ so that a term in $\mathbf{z}^*\mathbf{z}$ is unnecessary. The coefficients A, B, and C may now be determined to make $\beta^{(2)}$ left regular. Of course $\beta^{(2)}$ can always be multipled by an overall scalar constant without destroying the regularity so we arbitrarily choose to normalize $\beta^{(2)}$ so that when $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$, $|\beta^{(2)}|^2 = 1$ just as do $|\beta^{(1)}|^2$ and $|\beta^{(-1)}|^2$. When all this is done we obtain,

$$\beta^{(2)} = \frac{5}{8}z^2 + \frac{1}{4}zz^* + \frac{1}{8}z^{*2} \tag{44}$$

or,

$$\beta^{(2)} = x_3^2 - \frac{x_1^2+x_2^2}{2} + x_2x_3\hat{i}_1 - x_1x_3\hat{i}_2 \tag{45}$$

The above procedure may be continued indefinitely to yield all $\beta^{(n)}$'s for positive integral n. We now note two interesting properties of the $\beta^{(n)}$'s. First, they all commute with each other and second,

$$\frac{d}{d(-x_3)}\beta^{(n)} = n\beta^{(n-1)} \tag{46}$$

Equation (46) may be used to generate $\beta^{(-2)}$ from equations (41) and $\beta^{(-3)}$ from $\beta^{(-2)}$, etc. Upon doing this we find that the resulting $\beta^{(-n)}$'s are all left regular and mutually commuting. Using (46) we also find that $\beta^{(0)} = 1$.

The β's derived above are all quaternion potentials for three dimensional static fields. Their scalar parts being real, are just the ordinary electrostatic scalar potential. Writing these scalar parts in spherical coordinates we find,

$$\begin{array}{l} \vdots \\ \phi_0^{(n)} = (-1)^n r^n P_n(\cos\theta) \\ \vdots \\ \phi_0^{(2)} = r^2 P_2(\cos\theta) \\ \phi_0^{(1)} = -rP_1(\cos\theta) \end{array}$$

$$\phi_0^{(0)} = P_0(\cos\,\theta)$$

$$\phi_0^{(-1)} = -\frac{1}{r^2}P_1(\cos\,\theta)$$

$$\phi_0^{(-2)} = \frac{1}{r^3}P_2(\cos\,\theta) \tag{47}$$

$$\vdots$$

$$\phi_0^{(-n)} = (-1)^n\frac{1}{r^{n+1}}P_n(\cos\,\theta)$$

$$\vdots$$

where $P_\ell(\cos\,\theta)$ is the ℓ^{th} order Legendre polynomial and θ is the polar angle with the polar axis in the $\hat{i}_3$ direction. These are quite obviously the solutions of Laplace's equation in three dimensions as one might have expected. Note that the potential $\beta^{(-1)}$ corresponding to the dipole field represents a simple pole singularity in three dimensions (See Appendix A.)

A More General Method of Generating Potentials

Rudolf Fueter developed a more general method of obtaining left regular functions. By means of this method it will be possible to obtain a familiar time dependent solution of Maxwell's equations. Following Fueter[7] we proceed as follows.

Consider regular quaternion functions which depend on x_1, x_2, and x_3 only through $r = \sqrt{x_1^2+x_2^2+x_3^2}$ and $\hat{r} = \frac{\vec{r}}{r}$ where $\vec{r} = x_1\hat{i}_1 + x_2\hat{i}_2 + x_3\hat{i}_3$. We term such functions regular spherical functions*. We now appeal to a theorem due to Fueter which may be stated as follows.

> Let $w = u + iv = w(z)$ be an analytic function of the complex variable $z = x + i\,y$ where u and v are each real functions of x and y. Make the substitutions $x \to x_0$, $y \to r$, and $i \to \hat{r}$ so that $z \to x_0 + \vec{r} = \mathbf{x}$ and $w \to \mathbf{V} = U(x_0,r) + \hat{r}V(x_0,r) = \mathbf{V}(\mathbf{x})$. If we form $-\mathbf{v} \overset{\Omega}{=} \tfrac{1}{2}\Delta\mathbf{V}$ where Δ is the Laplacian in four dimensions, then $\mathbf{v}$ is fully regular and each component of $\mathbf{V}$ satisfies the four dimensional biharmonic equation $\Delta\Delta f = 0$.

By means of this theorem then one can generate a fully regular quaternion function $\mathbf{v}$ from any analytic function w of a complex variable z.

A simpler formula for $\mathbf{v}$ may be obtained in the following way.

$$\Delta\mathbf{V} = \left(\frac{\partial^2}{\partial x_0^2} + \nabla^2\right)\mathbf{V} = \frac{\partial^2 U}{\partial x_0^2} + \hat{r}\,\frac{\partial^2 V}{\partial x_0^2} + \nabla^2 U + \nabla^2(\hat{r}V) \tag{48}$$

but

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} \tag{49}$$

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*Note that a regular spherical function must be fully regular since a function dependent only on r can have no curl and Equations (20) and (21) are identical in that case.

and

$$\nabla^2(\hat{r}V) = \hat{r}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)V \tag{49}$$

Now $\frac{\partial^2 V}{\partial x_0^2} + \frac{\partial^2 V}{\partial r^2} = 0$ and $\frac{\partial^2 U}{\partial x_0^2} + \frac{\partial^2 U}{\partial r^2} = 0$ by analyticity of the generating function $w(z)$. Thus

$$\Delta\mathbf{V} = \frac{2}{r}\frac{\partial U}{\partial r} + \hat{r}\,\frac{2}{r}\frac{\partial V}{\partial r} - \frac{2}{r^2}\,\hat{r}V \tag{50}$$

$$\tfrac{1}{2}\Delta\mathbf{V} = \frac{1}{r}\frac{\partial U}{\partial r} + \hat{r}\left(\frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2}\right) = \frac{1}{r}\frac{\partial U}{\partial r} + \hat{r}\,\frac{\partial}{\partial r}\left(\frac{V}{r}\right) \tag{51}$$

The Cauchy-Riemann equations require that

$$\frac{\partial U}{\partial r} = -\frac{\partial V}{\partial x_0}$$

so that we have,

$$\tfrac{1}{2}\Delta\mathbf{V} = -\frac{1}{r}\frac{\partial V}{\partial x_0} + \hat{r}\,\frac{\partial}{\partial r}\left(\frac{V}{r}\right) = -\,\Pi\left(\frac{V}{r}\right) \tag{52}$$

Thus,

$$\mathbf{v} = \Pi\left(\frac{V}{r}\right) \tag{53}$$

and of course,

$$\Pi * \mathbf{v} = \Pi * \Pi\left(\frac{V}{r}\right) = \Delta\left(\frac{V}{r}\right) = \frac{1}{r}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial r^2}\right)V = 0 \tag{54}$$

(since V is harmonic in x_0 and r) implying that $\mathbf{v}$ is left regular and since it is also spherical it is fully regular. Evidently Equation (54) also holds for $U(x_0,r)$ and we thus have another fully regular function

$$\mathbf{u} = \Pi\left(\frac{U}{r}\right) \tag{55}$$

We term the functions $\mathfrak{u}$ and $\mathfrak{v}$ twin functions as they are obtained from the same analytic function w(z).

Now, given any regular spherical function

$$\mathfrak{u} = \phi(x_0,r) + \hat{r}\,\psi(x_0,r) \tag{56}$$

we can always find a $U(x_0,r)$ and a $V(x_0,r)$ such that

$$\frac{1}{r}\frac{\partial U}{\partial r} = \phi \qquad \frac{\partial}{\partial r}\left(\frac{V}{r}\right) = \psi \tag{57}$$

$$\frac{\partial U}{\partial x_0} = \frac{\partial V}{\partial r} \qquad \frac{\partial V}{\partial x_0} = -\frac{\partial U}{\partial r} \tag{58}$$

by means of

$$U = \int r\,\phi\,dr + f(x_0) \tag{59}$$

$$V = r\int \psi\,dr + r\,g(x_0) \tag{60}$$

where $f(x_0)$ and $g(x_0)$ are determined to within additive constants by (58). Thus we can always find an analytic function of a complex variable which will generate $\mathfrak{u}$. What we have found then is that every regular spherical quaternion function $\mathfrak{u}$ has a scalar potential $\frac{U}{r}$. One can also show that $\mathfrak{u}$ has a quaternic potential $\dot{\mathfrak{u}}$. (See Appendix B.)

Examples of Regular Spherical Functions

We now present a few simple examples of regular functions generated from analytic functions of a complex variable

Example 1: $w(z) = a\,\ell n\, z = \frac{a}{2}\,\ell n(x^2+y^2) + i\,a\,\tan^{-1}\frac{y}{x}$

$$\mathbf{u} = \frac{a}{r}\,\frac{x_0}{x_0^2+r^2} + \hat{r}\,a\left[\frac{1}{r^2}\,\ell n\sqrt{x^2+y^2} - \frac{1}{x_0^2+r^2}\right]$$

$$\mathbf{v} = \frac{a}{x_0^2+r^2} + \hat{r}\,a\left[\frac{x_0}{r(x_0^2+r^2)} - \frac{1}{r^2}\tan^{-1}\frac{r}{x_0}\right]$$

Example 2: $w(z) = \frac{a}{z} = a\,\frac{x-iy}{x^2+y^2}$

$$\mathbf{u} = -\,\frac{a}{r}\,\frac{r^2-x_0^2}{(r^2+x_0^2)^2} + \hat{r}\,a\,\frac{x_0}{r^2+x_0^2}\left[\frac{2}{r^2+x_0^2} + \frac{1}{r^2(r^2+x_0^2)}\right]$$

$$\mathbf{v} = \frac{2a(x_0-r\hat{r})}{(x_0^2+r^2)^2}$$

Example 3: $w(z) = az = ax + iay$

$$\mathbf{u} = \frac{a}{r} + \hat{r}\,\frac{ax_0}{r^2} \qquad\qquad \mathbf{v} = 0$$

Example 4: $w(z) = a + ib$

$$\mathbf{u} = \frac{a\hat{r}}{r^2} \qquad\qquad \mathbf{v} = \frac{\hat{r}b}{r^2}$$

Note that these and all regular spherical functions are fully regular and hence cannot generate fields.

The Time Harmonic Electric Dipole Field

The above formalism will now be used to derive a quaternion potential for the fields of an oscillating electric dipole. We begin with the analytic function

$$w(z) = \frac{a}{k^2} \cosh kz \tag{61}$$

Then,

$$\mathbf{u} = \frac{a}{kr} \cos kr \sinh kx_0 + a\hat{r}[\frac{1}{kr} \sin kr + \frac{1}{(kr)^2} \cos kr]\cosh kx_0 \tag{62}$$

$$\mathbf{v} = \frac{a}{kr} \sin kr \cosh kx_0 + a\hat{r}[- \frac{1}{kr} \cos kr + \frac{1}{(kr)^2} \sin kr]\sinh kx_0 \tag{63}$$

Making the substitution $x_0 \to i\ t$ we obtain,

$$\mathbf{u} \to \frac{ia}{kr} \cosh kr \sin \omega t + \hat{r}a[\frac{1}{kr} \sin kr + \frac{1}{(kr)^2} \cos kr]\cos \omega t \tag{64}$$

$$\mathbf{v} \to \frac{a}{kr} \sin kr \cos \omega t + ai\hat{r}[\frac{-1}{kr} \cos kr + \frac{1}{(kr)^2} \sin kr]\sin \omega t \tag{65}$$

We now form $\mathbf{w} = -\mathbf{u} + i\mathbf{v}$

$$\mathbf{w} = \frac{ia}{kr} \sin(kr-\omega t) - \hat{r}a[\frac{1}{kr} \sin(kr-\omega t) + \frac{1}{(kr)^2} \cos(kr-\omega t)] \tag{66}$$

Let $\hat{e}$ be a unit constant vector, a constant quaternion with zero scalar part. Multiplying $\mathbf{w}$ on the right by $\hat{e}$ will not destroy its left regularity but will destroy its right regularity by imparting to it a curl and enabling it to yield fields.

$$\nabla \hat{e} = i\,\hat{e}\,a\left[\frac{1}{kr}\sin(kr-\omega t)\right] + a(\hat{r}\cdot\hat{e})\left[\frac{1}{kr}\sin(kr-\omega t) + \frac{1}{(kr)^2}\cos(kr-\omega t)\right]$$

$$- a(\hat{r}\times\hat{e})\left[\frac{1}{kr}\sin(kr-\omega t) + \frac{1}{(kr)^2}\cos(kr-\omega t)\right]$$

$$= \boldsymbol{\phi} + i\mathbf{A} \tag{67}$$

Thus, selecting a spherical coordinate system with its polar axis in the direction of $\hat{e}$, we obtain

$$\boldsymbol{\phi} = \left[\frac{a}{kr}\sin(kr-\omega t) + \frac{a}{(kr)^2}\cos(kr-\omega t)\right]\cos\theta$$

$$+ \hat{i}_\phi\left[\frac{a}{kr}\sin(kr-\omega t) + \frac{a}{(kr)^2}\cos(kr-\omega t)\right]\sin\theta \tag{68}$$

$$\mathbf{A} = \hat{i}_r\left[\frac{a}{kr}\sin(kr-\omega t)\right]\cos\theta - \hat{i}_\theta\left[\frac{1}{kr}\sin(kr-\omega t)\right]\sin\theta \tag{69}$$

Finally, we obtain the fields by means of (29) and (30),

$$\vec{E} = \nabla\times\vec{\phi} = 2a\,k\cos\theta\left[\frac{1}{(kr)^2}\sin(kr-\omega t) + \frac{1}{(kr)^3}\cos(kr-\omega t)\right]\hat{i}_r$$

$$+ a\,k\sin\theta\left[\frac{1}{(kr)^2}\sin(kr-\omega t) - \left(\frac{1}{kr} - \frac{1}{(kr)^3}\right)\cos(kr-\omega t)\right]\hat{i}_\theta \tag{70}$$

$$\vec{B} = \nabla\times\vec{A} = a\,k\sin\theta\left[\frac{1}{(kr)^2}\sin(kr-\omega t) - \frac{1}{kr}\cos(kr-\omega t)\right]\hat{i}_\phi \tag{71}$$

These are easily recognized as the electric dipole fields as anticipated. (The dipole moment p is proportional to the constant a.)

Concluding Remarks

We have seen that the quaternion formalism enables one to obtain in an essentially algebraic way some of the most fundamental results of electromagnetic theory. One might hope that, being so analogous to complex variable theory, this formalism would provide four dimensional analogs of conformal mapping, residue theory, analytic continuation, and so on. However, it is well known among mathematicians that a full analytic theory in the complex variable sense is not possible for quaternions. In particular, conformal mapping in more than two dimensions cannot be done. (In our case, this is because a regular function of a regular function is not necessarily regular.) Certain of the complex variable results, however do have analogs in four dimensions. For example, Fueter has provided analogs of Cauchy's theorem, Morera's theorem, and power series expansions. Moreover, the possibility of a four dimensional theory of analytic continuation has not been ruled out. Future application of all of these aspects of the quaternion calculus to electromagnetics might prove quite interesting from both the theoretical (elegance) and the practical (calculational techniques) points of view.

In a more general sense, Edmonds has pointed out that quaternions may provide the basis for future fundamental breakthroughs in several areas of theoretical physics.[17]

Appendix A - Singularities

The fundamental singularity in complex variable theory is the simple pole; e.g., $\frac{1}{w}$ where w is a complex variable. We have shown above that the simple pole singularity in three dimensions is $\boldsymbol{\beta}^{(-1)}$ which may be written $\frac{1}{\mathbf{z}|\mathbf{z}|}$ where $\mathbf{z}$ is $-x_3-x_2\hat{i}_1+x_1\hat{i}_2$. If a procedure similar to the one used to generate the $\boldsymbol{\beta}$'s is applied to the full four component quaternion $\mathbf{q}$ rather than the three component quaternion $\mathbf{z}$, one finds that the simple pole singularity is $\frac{1}{\mathbf{q}|\mathbf{q}|^2}$. The corresponding singularity in one dimension is $\frac{|x|}{x}$ where x is a real variable. Thus we generate a sequence of simple pole type singularities each corresponding to a space of a certain dimensionality. If we take v to be the variable of corresponding dimensionality, i.e., real, complex, three component quaternion, four component quaternion for one, two, three, and four dimensions respectively, we obtain the sequence,

$$\frac{|v|}{v}\ ,\ \frac{1}{v}\ ,\ \frac{1}{v|v|}\ ,\ \frac{1}{v|v|^2}$$

Appendix B - Quaternic Potentials

Consider a left regular function $\mathbf{u}$ and define

$$\dot{\mathbf{u}} = \tfrac{1}{2}\Pi\,\mathbf{u}$$

Now,

$$\Pi\,{}^*\dot{\mathbf{u}} = \tfrac{1}{2}\Pi\,{}^*\Pi\,\mathbf{u} = \tfrac{1}{2}\Pi\,\Pi\,{}^*\mathbf{u} = 0$$

that is $\dot{\mathbf{u}}$ is also left regular. Similarly, if $\mathbf{u}$ had been right regular, then $\dot{\mathbf{u}}$ would have been right regular. If $\mathbf{u}$ had been fully regular then $\dot{\mathbf{u}}$ would have been fully regular. Thus if $\mathbf{u}$ is a regular spherical function then $\dot{\mathbf{u}}$ is also. We term $\mathbf{u}$ the quaternic potential for $\dot{\mathbf{u}}$. In the case of regular spherical functions it can easily be shown that if $\mathbf{u}$ is generated from an analytic function $w(z)$ then $\dot{\mathbf{u}}$ is generated from $\frac{dw}{dz}$. Now, since analytic functions of a complex variable possess integrals (potentials) and derivatives of all orders, we have that regular spherical functions of a quaternion variable possess potentials and derivatives of all orders.

Recall that we have shown previously that every regular spherical function possesses a scalar potential and just above we found that it possesses a quaternic potential. These two potentials are related. From (20) and (21) we find that a fully regular function must satisfy,

$$\frac{\partial u_0}{\partial x_0} = \nabla\cdot\vec{u} \qquad \text{and} \qquad \frac{\partial\vec{u}}{\partial x_0} = -\nabla u_0$$

Now, by definition,

$$\Pi\, u_0 = \frac{\partial u_0}{\partial x_0} - \nabla u_0$$

Substitution shows that this may be written,

$$\Pi\, u_0 = \tfrac{1}{2}\left(\frac{\partial u_0}{\partial x_0} - \nabla u_0\right) + \tfrac{1}{2}\left(\nabla\cdot\vec{u} + \frac{\partial\vec{u}}{\partial x_0}\right) = \tfrac{1}{2}\Pi\,\mathbf{u} = \dot{\mathbf{u}}$$

Thus, the scalar potential for $\dot{\mathbf{u}}$ is just the scalar part of the quaternic potential.

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