Mathematics Notes

Note 35

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On the Use of Contour Integration for Finding Poles, Zeros, Saddles, and Other Function Values in the Singularity Expansion Method

Carl E. Baum
Air Force Weapons Laboratory

Abstract

Electromagnetic problems are analytic functions of the complex frequency s except at singularities. This analytic property allows one to apply contour integral theorems from complex variable theory. The argument number integral can be used to find the zeros and poles inside the contour. For cases of no singularities inside the contour the Cauchy integral formula can be used to find the zeros as well as the function values. If the function of interest requires lengthy computations (such as a determinant) then the Cauchy integral formula provides a convenient way to construct the function values from a numerical integration. This is basically an approximate analytical continuation procedure. Saddles of the function are also found by such procedures. These procedures provide convenient ways to display the function in contour plots in the s plane, suitable for synthesis procedures involving root locus techniques.

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I. Introduction

In the singularity expansion method one of the important types of terms is the natural frequencies. These are the poles of the response of an object such as an antenna or scatterer in the complex frequency plane. For objects of sufficiently simple shape, the natural frequencies may be found from the roots of some equation involving well known special functions.[1,5] In general, for arbitrarily shaped objects (even of finite dimensions) such special function techniques do not apply.

For general objects the electromagnetic response is given from the solution of an integral equation which can be written in the form

$$\langle \tilde{\vec{\Gamma}}(s); \tilde{\vec{U}}(s) \rangle = \tilde{\vec{I}}(s) \tag{1.1}$$

where $\tilde{\vec{U}}(s)$ is a normalized response for the current to a delta function source (in time) $\tilde{\vec{I}}(s)$ and where the domain of integration is over that portion of the object where the response exists.[7] In finding the natural frequencies either of the following equations is needed

$$\langle \tilde{\vec{\Gamma}}(s_\alpha); \vec{\nu}_\alpha \rangle = \vec{0}$$

$$\langle \vec{\mu}_\alpha; \tilde{\vec{\Gamma}}(s_\alpha) \rangle = \vec{0} \tag{1.2}$$

The solution of these equations gives the natural frequencies, s_α, which are special values of the complex frequency where the response $\tilde{\vec{U}}(\vec{r},s)$ has poles. Here we exclude any branch cuts from the region of the complex frequency plane of interest. In many cases of interest there are no branch cuts in the entire s plane.

The coupling coefficients for first order poles are

$$\tilde{\eta}_\alpha(s_\alpha) = \frac{\langle \vec{\mu}_\alpha; \tilde{\vec{I}}(s_\alpha) \rangle}{\langle \vec{\mu}_\alpha; \tilde{\vec{\Gamma}}_1(s_\alpha); \vec{\nu}_\alpha \rangle} \tag{1.3}$$

where

$$\tilde{\vec{\Gamma}}_1(\vec{r},\vec{r}';s) \equiv \frac{\partial}{\partial s}\tilde{\vec{\Gamma}}(\vec{r},\vec{r}';s) \tag{1.4}$$

This gives terms in the delta function response as

$$\tilde{\vec{U}}(\vec{r},s) = \sum_\alpha \tilde{n}_\alpha(s)\vec{v}_\alpha(\vec{r})(s - s_\alpha)^{-n_\alpha}$$

$$+ \text{ possible entire function} \tag{1.5}$$

where n_α is typically 1. For $n_\alpha > 1$ at any particular s_α the formulas for the coupling coefficients are more complicated than those above.

For the more general object (antenna, scatterer, etc.) shapes one can use the common moment methods[17] to convert an integral equation to a matrix equation of the form[5]

$$(\tilde{\Gamma}_{n,m}(s)) \cdot (\tilde{U}_m(s)) = (\tilde{I}_n(s))$$

$$n,m = 1, 2, \cdots, N \tag{1.6}$$

The choice of some positive integer N determines the numbers of components of the excitation and response vectors and of the matrix (N × N) relating the two. The N components of the vectors may represent samples of the physical three component vectors at various localized zones of the object (point matching) or may represent coefficients of more general function expansions (usually orthogonal functions) of the excitation and response functions.

The integral equation 1.1 is then replaced for general objects by its numerical approximation, the matrix equation 1.6, with solution

$$(\tilde{U}_m(s)) = (\tilde{\Gamma}_{n,m}(s))^{-1} \cdot (\tilde{I}_n(s)) \tag{1.7}$$

provided

$$\det(\tilde{\Gamma}_{n,m}(s)) \neq 0 \tag{1.8}$$

Provided the moment method approximation of the integral equation is performed in a manner that retains the analytic properties of the operator as a function of s, then equations 1.2 through 1.5 carry over respectively as

$$(\tilde{\Gamma}_{n,m}(s_\alpha)) \cdot (\nu_m)_\alpha = (0_n)$$

$$(\mu_n)_\alpha \cdot (\tilde{\Gamma}_{n,m}(s_\alpha)) = (0_n)$$

$$\tilde{\eta}_\alpha(s_\alpha) = \frac{(\mu_n)_\alpha \cdot (\tilde{I}_n(s_\alpha))}{(\mu_n)_\alpha \cdot (\tilde{\Gamma}_{n,m}(s_\alpha))_1 \cdot (\nu_m)_\alpha} \tag{1.9}$$

$$(\tilde{\Gamma}_{n,m}(s))_1 \equiv \frac{d}{ds} (\tilde{\Gamma}_{n,m}(s))$$

$$(\tilde{U}_m(s)) = \sum_\alpha \tilde{\eta}_\alpha(s) (\nu_m)_\alpha (s - s_\alpha)^{-n_\alpha}$$

where n_α is typically 1. Actually the natural frequencies, natural modes, and coupling coefficients in equations 9.1 are only approximations of the exact values from the previous operator equations. A good moment method approximation scheme will have the approximate natural frequencies, natural modes, and coupling coefficients approach the exact ones closely for sufficiently large N. Note that (0_n) is an N component zero vector and $(0_{n,m})$ is an N × N zero matrix.

As is well known the solutions of the first two of equations 1.9 occur for

$$\det(\tilde{\Gamma}_{n,m}(s_\alpha)) = 0 \tag{1.10}$$

which serves as an equation for the natural frequencies, s_α, which can be determined numerically. Note that α is in general some convenient set of indices for the roots so determined.

The determinant is an analytic function of s provided the matrix elements are analytic functions of s. Let us then define a function

$$\tilde{d}(s) \equiv \det(\tilde{\Gamma}_{n,m}(s)) \tag{1.11}$$

where this definition applies for problems cast in the form of equation 1.7. This function $\tilde{d}(s)$ can also be considered as a function corresponding to the operator equivalent of matrix determinant for equation 1.1. In general terms $\tilde{d}(s)$ can be called the denominator function in the sense of excitation (or something related to excitation) "divided" by $\tilde{d}(s)$. This more general definition of $\tilde{d}(s)$ applies to various closed form solutions involving separable or partially separable solutions for special coordinate systems, such as spherical coordinates where $\tilde{d}(s)$ involves spherical Hankel functions.[5] This brings up other possible forms of $\tilde{d}(s)$ such as the eigenvalues of the operator or matrix equation. This will hopefully be included in a future note on the relation of eigenfunction expansions and SEM.

Having defined a $\tilde{d}(s)$ for the problem at hand the first question to be addressed is to find the poles of the response which are the zeros of $\tilde{d}(s)$ as

$$\tilde{d}(s_\alpha) = 0 \tag{1.12}$$

If $\tilde{\tilde{I}}(s)$ corresponds to a delta function excitation (in time domain) then $\tilde{\tilde{I}}(s)$ is typically an entire function of s and all response poles are zeros of $\tilde{d}(s)$. There may also be poles of $\tilde{d}(s)$ such as at $s = 0$ due to the form of the dyadic Green's function, but these may be removed if desired.

Now $\tilde{d}(s)$ may take much computer time to calculate if it is a determinant or a function of a determinant. Finding the zeros of $\tilde{d}(s)$ by calculations of $\tilde{d}(s)$ and including special root searching procedures involving various s may may be time consuming due to all the determinants to be calculated. This note discusses procedures for finding the natural frequencies while significantly reducing the number of values of $\tilde{d}(s)$ which must be calculated from its original definition, such as equation 1.11.

Consider the complex frequency plane, $s = \Omega + i\omega$, as in figure 1.1. Let C be a closed directed curve which does not cross or touch itself. As shown in figure 1.1 this curve is a contour with counterclockwise direction for purpose of integrals over C in the complex s plane. This note is concerned with using the values of $\tilde{d}(s)$ (or other function) analytic on C to determine values and other characteristics of d(s) inside C, i.e., for $s \in A$ where A is the open region enclosed by C.

Now consider M points $s = s'_m$ for $m = 1, \cdots, M$ where all s'_m are on the contour, i.e., $s'_m \in C$ for $m = 1, \cdots, M$.

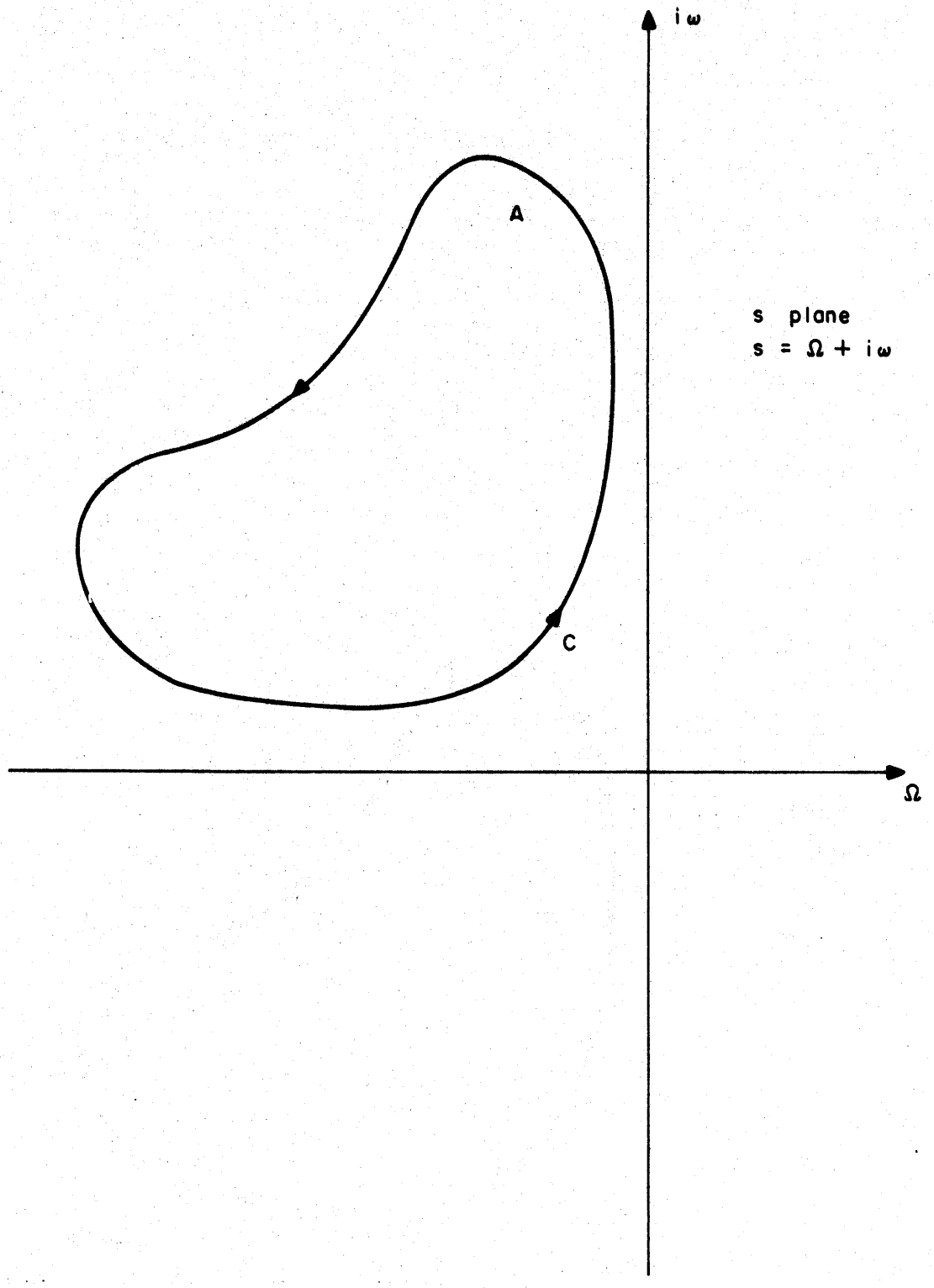


Figure 1.1 Contour for Finding Poles and Zeros in the Complex Frequency Plane

Furthermore let the s'_m be arranged in ascending order of m for positive direction along C (counterclockwise). For convenience in the progressions and sums we can let

$$s'_m = s'_{m+M} = s'_{m-M} \tag{1.13}$$

so that the set of points s'_m can be considered cyclic. Note that among the set of points s'_m the choice of which corresponds to m = 1 is arbitrary. Once this choice is fixed, however, all the remaining indices m for s'_m are determined. Typically the points s'_m would be roughly uniformly spaced around C.

From M calculated values of $\tilde{d}(s)$ on C (i.e., $\tilde{d}(s'_m)$) one can approximately calculate the values of d(s) inside C (i.e., for $s \in A$). In particular the zeros and poles of d(s) inside C can be approximately calculated. Suppose d(s) is lengthy to calculate from some defining relation such as a determinant. Then using some simpler formulas discussed in this note numerous values as well as zeros and poles of $\tilde{d}(s)$ inside C can be more quickly determined (at least approximately) using once calculated (and stored in memory) values of $\tilde{d}(s)$ on the contour C (i.e., $\tilde{d}(s'_m)$).

II. Determining the Zeros and Poles Via the Argument Number

Consider a function f(s) which is analytic and non zero on the contour C with only zeros and poles inside C (i.e., $s \in A$). Then we have the principle of the argument, or what might be referred to as the argument number N_a, as[11,14,15]

$$N_a \equiv \frac{1}{2\pi i} \oint_C \frac{1}{\tilde{f}(s')} \frac{d}{ds'} \tilde{f}(s')ds' \tag{2.1}$$

where

$$N_a = N_o - N_p \equiv \text{argument number}$$

$$N_o \equiv \text{zero number}$$

$$= \text{number of zeros inside C including multiplicity} \tag{2.2}$$

$$N_p \equiv \text{pole number}$$

$$= \text{number of poles inside C including multiplicity}$$

$$s' = \text{dummy variable for values of s on C}$$

The zero and pole numbers can be expressed as

$$N_o = \sum_A n_{o_j}$$

$$n_{o_j} = \text{multiplicity of jth zero in A} \tag{2.3}$$

$$N_p = \sum_A n_{p_k}$$

$$n_{p_k} = \text{multiplicity of kth pole in A}$$

Note that equations 2.1 through 2.3 are valid for simple closed contour C (nowhere crossing itself) taken in the positive direction (counterclockwise). If more general closed

contours are used then one must not use N_o and N_p but express the argument number as

$$N_a = \sum_j n_{w_{o_j}} n_{o_j} - \sum_k n_{w_{p_k}} n_{p_k}$$

$n_{w_{o_j}}$ = winding number of C with respect to the jth zero

$n_{w_{p_k}}$ = winding number of C with respect to the kth pole (2.4)

The winding number of C with respect to a point (zero, pole, or otherwise) is the number of times that C goes around that point in the positive direction.[16] Note that a winding number can be negative or zero. For simple closed positive directed C the winding number for any s in A is +1, reducing equations 2.4 to the simpler case in equations 2.1 through 2.3; it is this simpler case which is our primary concern.

There are several forms in which to express the argument number. Starting from equation 2.1 we have

$$N_a = \frac{1}{2\pi i} \oint_C \frac{1}{\tilde{f}(s')} \frac{d}{ds'} \tilde{f}(s')ds'$$

$$= \frac{1}{2\pi i} \oint_C \frac{d(\tilde{f}(s'))}{\tilde{f}(s')}$$

$$= \frac{1}{2\pi i} \oint_C \frac{d}{ds'}[\ell n(\tilde{f}(s'))]ds$$

$$= \frac{1}{2\pi i} \oint_C d[\ell n(\tilde{f}(s'))]$$

$$= \frac{1}{2\pi i} \ell n(\tilde{f}(s')) \Big|_C \oint$$

$$= \frac{1}{2\pi i}[\ell n(|\tilde{f}(s')|) + i \arg(\tilde{f}(s'))] \oint_C$$

$$= \frac{1}{2\pi} \arg(\tilde{f}(s')) \oint_C \qquad (2.5)$$

A new symbol has been introduced for multivalued functions on C as

$$\tilde{F}(s') \oint_C \equiv \text{change in the function } \tilde{F}(s') \text{ around the contour C in the positive sense} \qquad (2.6)$$

For a function such as arg(f(s)) which is not in general a single valued function on C this change is not in general zero. Starting at one point s* on C choose one of the values of the function $\tilde{F}(s^*)$. Moving around C in the positive sense while constraining the values of $\tilde{F}(s)$ to vary continuously from $\tilde{F}(s^*)$ calculate the value of $\tilde{F}(s)$ at the "end" of the contour C back at s* which we can call $\tilde{F}(s^*-)$. Then we have

$$\tilde{F}(s) \oint_C = \tilde{F}(s^*-) - \tilde{F}(s^*) \qquad (2.7)$$

The result is independent of the choice of s* as long as $s^* \in C$, and independent of whichever of the possible values of $\tilde{F}(s^*)$ that is chosen.

In computing N_a numerically around a contour C one should note that the complex argument function $\arg(\tilde{f}(s))$ is multivalued and that for computer work a principal value is chosen (such as ATAN2 on the CDC 6600 defined to give $-\pi < \arg(\tilde{f}(s)) < \pi$). As illustrated in figure 2.1 one merely needs to count each jump from $+\pi$ to $-\pi$ as +1 and each jump from $-\pi$ to $+\pi$ as -1 and sum up these jump values around C in the positive sense to obtain N_a. Figure 2.1 shows a case of 4 jumps from $+\pi$ to $-\pi$ and 1 jump from $-\pi$ to $+\pi$ giving $N_a = 3$. Such a diagram as in figure 2.1 can be helpful in keeping track of the argument around the contour.

As indicated in section I there are M points s'_m chosen on C, roughly uniformly spaced around C. One does not have, then, a continuous $\arg(\tilde{f}(s))$ around C, but only sampled values $\arg(\tilde{f}(s'_m))$. Between two successive points s'_m and s'_{m+1} we have

$$\Delta_m[\arg(\tilde{f}(s'))] \equiv \arg(\tilde{f}(s'_{m+1})) - \arg(\tilde{f}(s'_m)) \qquad (2.8)$$

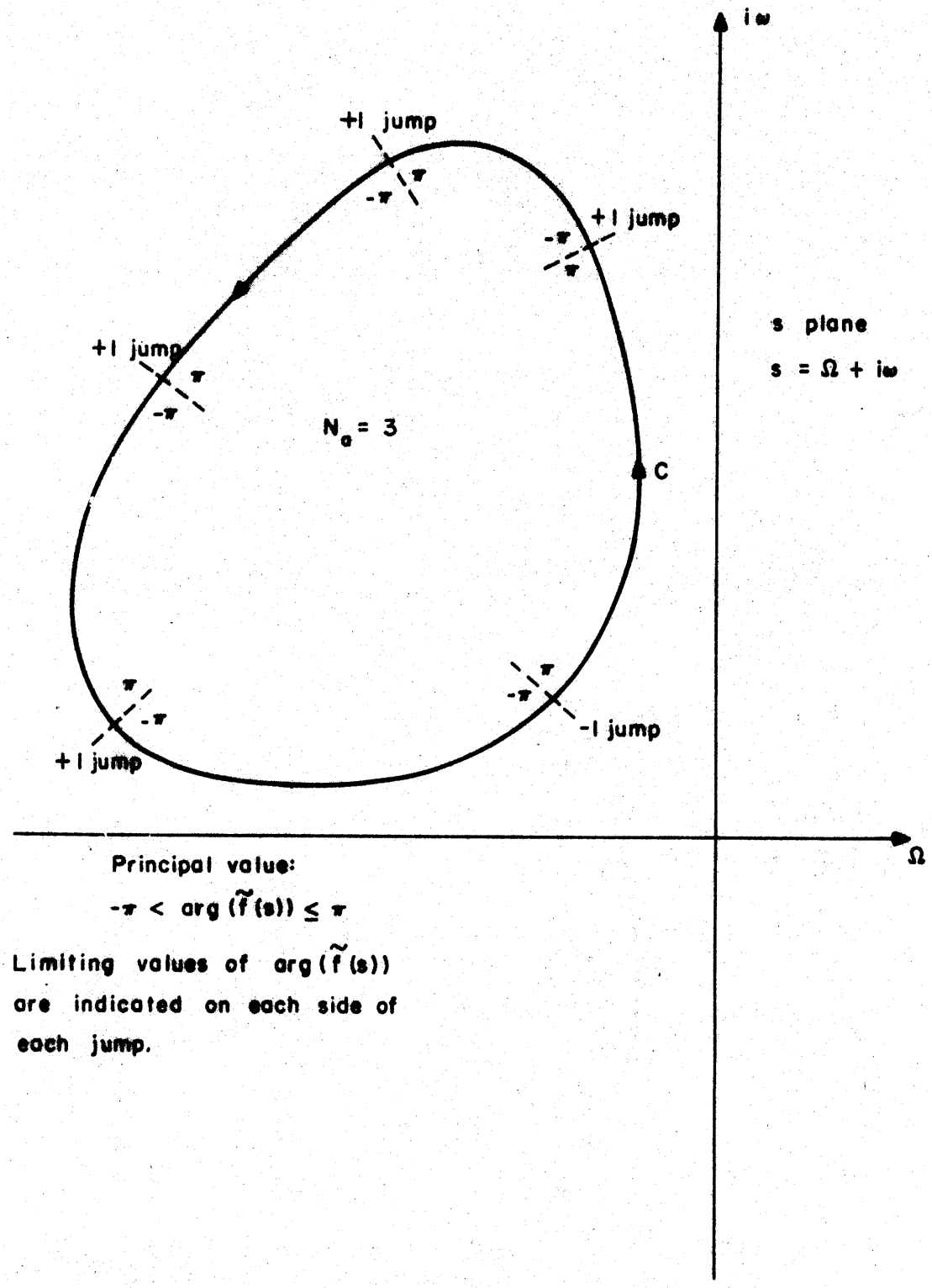


Figure 2.1 Example of Computing the Argument Number from the Principal Values of the Argument on the Contour

If

$$|\Delta_m[\arg(\tilde{f}(s'))]| << \pi$$

with an appropriate principal value definition for $\arg(\tilde{f}(s'))$ then one would normally expect that there was no jump in $\arg(\tilde{f}(s'))$ which was some integer multiple of 2π in going from s'_m to s'_{m+1} on the contour C. If the change in $\arg(\tilde{f}(s'))$ is large in going from s'_m to s'_{m+1} then the contour C is near one or more zeros and/or poles at this part of the contour C and a finer gridding of the sample points s'_m is needed in this region so that the change in $\arg(\tilde{f}(s))$ between sample points is made small. Note that $\arg(\tilde{f}(s'_m))$ near $\pm\pi$ is considered close to $\arg(\tilde{f}(s'_{m+1}))$ near $\mp\pi$ when principal value $-\pi < \arg(\tilde{f}(s)) \leq \pi$ is used.

Viewed another way if one is given M sample points s'_m from the contour C there is an infinity of contours C' which pass through these same points. If a zero or pole is near C as compared to the local spacing between the s'_m then an alternate contour C' could "easily" have passed on the opposite side of that zero or pole. The values of N_a for C and C' would then differ by the N_a associated with that particular zero or pole. Alternatively C' may pass through the zero or pole giving a non integer value to N_a if the integral through the zero or pole is taken in an appropriate principal value sense.

As the above discussion indicates some care must be exercised in the numerical evaluation of the argument number around C. Numerical analysis of the errors in such numerical contour integrals may give better quantitative restrictions on the spacing of the s'_m.

An alternative approach to establishing the numerical accuracy of a determination of N_a consists of numerically integrating the contour integral. The previously discussed approach relied on choosing the proper branch of a multivalued function which gave a closed form of the indefinite integral on the contour C.

In this alternative approach one can consider the various integral forms for N_a as given in equation 2.5. In particular for present purposes let us choose

$$N_a = \frac{1}{2\pi i} \oint_C \frac{d(\tilde{f}(s'))}{\tilde{f}(s')}$$

$$\simeq \frac{1}{2\pi i} \oint_C \frac{\Delta[\tilde{f}(s')]}{\tilde{f}(s')} \tag{2.9}$$

where this last summation notation is a direct carryover from the closed contour integral notation. This type of summation might logically be referred to as a closed contour sum. There are various explicit forms such a contour sum may take, depending on how $\Delta[\tilde{f}(s')]$ is defined and how $\tilde{f}(s')$ is evaluated. The choices involve both the choice of the discrete s' (i.e. the s'_m) and the manner in which $\Delta[\tilde{f}(s')]$ and $\tilde{f}(s')$ are averaged over adjacent values of $f(s'_m)$. One choice has

$$\Delta[\tilde{f}(s')] = \tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)$$

$$\tilde{f}(s') \simeq \frac{1}{2}[\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)]$$

$$N_a \simeq \frac{1}{\pi i} \sum_{m=1}^{M} \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)} \tag{2.10}$$

$$s'_{M+1} \equiv s'_1$$

A second choice is

$$\Delta[\tilde{f}(s')] = \frac{1}{2}[\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})]$$

$$\tilde{f}(s') = \tilde{f}(s'_m)$$

$$N_a \simeq \frac{1}{4\pi i} \sum_{m=1}^{M} \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)} \tag{2.11}$$

$$s'_{M+1} \equiv s'_1$$

$$s'_0 \equiv s'_M$$

The first choice bases the individual terms in the sum on an average value between the values at s'_m and s'_{m+1}. The second

choice bases the individual terms in the sum on an average centered on s'_m.

Equations 2.10 and 2.11 have an advantage in calculating N_a in that there is no ambiguity resulting from the multivalued nature of $\arg(\tilde{f}(s))$. As $M \to \infty$ the value N_a calculated from such summations will converge to the true value. For cases that C does not intersect any zeros or poles (or any other singularities) N_a must be an integer. Thus a criterion for whether or not M has been chosen sufficiently large and whether or not the s'_m positions around the contour C have been optimally chosen is the convergence of the approximations for N_a to an integer, which is of course real. Note that the approximations for N_a give in general complex numbers. If this type of check is applied with two or more different approximation sums for N_a the results of the various approximations can also be compared to each other to see if adequate convergence has been obtained.

The sums in equations 2.10 and 2.11 are quite simple to calculate on a computer. For cases that $\tilde{f}(s)$ requires lengthy numerical computation (such as a determinant of a large matrix) then one would like to minimize M, the number of different values of $\tilde{f}(s)$ to be calculated. Note, however, that these values of $\tilde{f}(s)$, once calculated, can be stored and used in other numerical contour integrals for finding the zeros of $\tilde{f}(s)$ for $s \in A$. These other contour integrals are discussed in later sections.

Having found the value of N_a for a particular contour then one would like to locate each of the zeros and poles (and determine their order) of $f(s)$ for $s \in A$. One procedure for doing such would involve subdividing A, say into two parts with boundary contours C_1 and C_2 so that

$$N_a\big|_C = N_a\big|_{C_1} + N_a\big|_{C_2} \equiv N_{a_1} + N_{a_2} \tag{2.12}$$

For N_A areas this result becomes

$$N_a\big|_C = \sum_{n=1}^{N_A} N_a\big|_{C_n} \equiv \sum_{n=1}^{N_A} N_{a_n} \tag{2.13}$$

where C_n is the contour around the nth area A_n (in the positive sense) so that we have

$$A = \bigcup_{n=1}^{N_A} A_n \tag{2.14}$$

with the common boundaries included in the union. There is the general contour integral result

$$\oint_C = \sum_{n=1}^{N_A} \oint_{C_n} \tag{2.15}$$

for N_a or any other quantity resulting from an integrable function. This area subdivision is illustrated in figure 2.2 for an example case of $N_A = 3$.

By inspecting the values of N_{a_n} as subdivision of A proceeds one can more and more localize the zeros and poles by looking at the various N_{a_n} values. Suppose that the function of interest is a denominator function $d(s)$ as discussed in section I. Further assume that $\tilde{d}(s)$ is known to have no poles but perhaps zeros for $s \in A$. Determinants for antenna and scattering problems often have no poles except possibly at $s = 0$ in the finite complex frequency plane. Then N_a can only be a positive integer or zero. Similarly upon subdivision the N_{a_n} can only be positive integers or zero. Whenever a zero argument number is found the corresponding area (say A_n) is discarded. Whenever an argument number equal to 1 is found there is known to be precisely one first order zero in the corresponding area. If all subareas have argument numbers 0 or 1 then one can consider all the zeros localized, or at least separated (for perhaps subsequent accurate location of the zeros). If, however, there is a second (or higher) order zero or two (or more) closely spaced zeros then one may have difficulty determining the exact situation and be faced with an argument number of 2 or more as the area of concern is successively subdivided.

Thus the argument number is useful for determining how many zeros are in A and for approximately localizing these; this procedure applies to functions with only zeros in A. By a simple extension this same procedure applies equally well to finding and approximately localizing the poles of a function with only poles in A. For a more general function with both zeros and poles in A finding and separating the zeros and poles can be more difficult, particularly if a zero and a pole are located near to each other.

Another situation where the argument number is useful is the case that a zero or pole is found but the order of the zero or pole is in question. Then a simple application of the

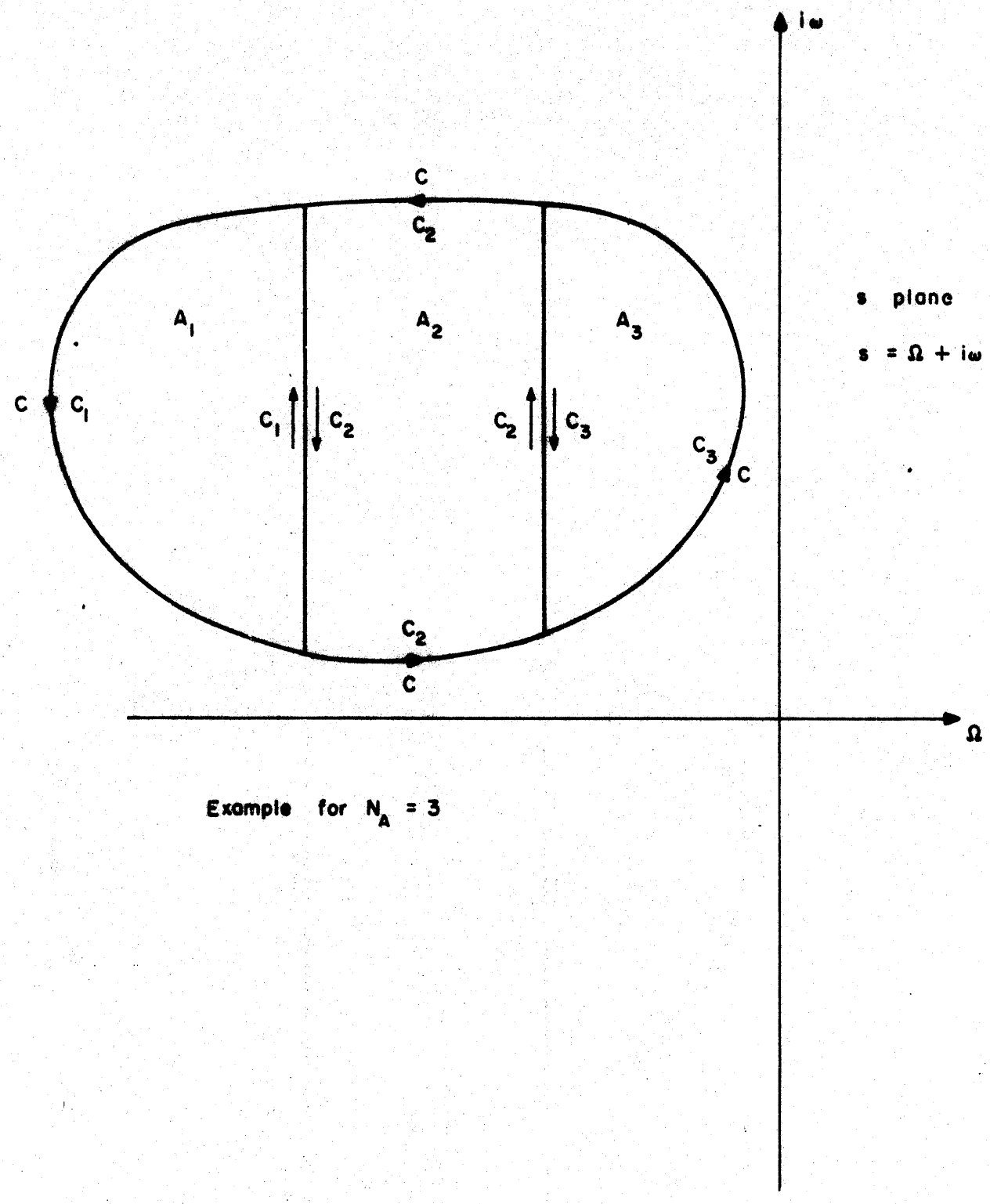


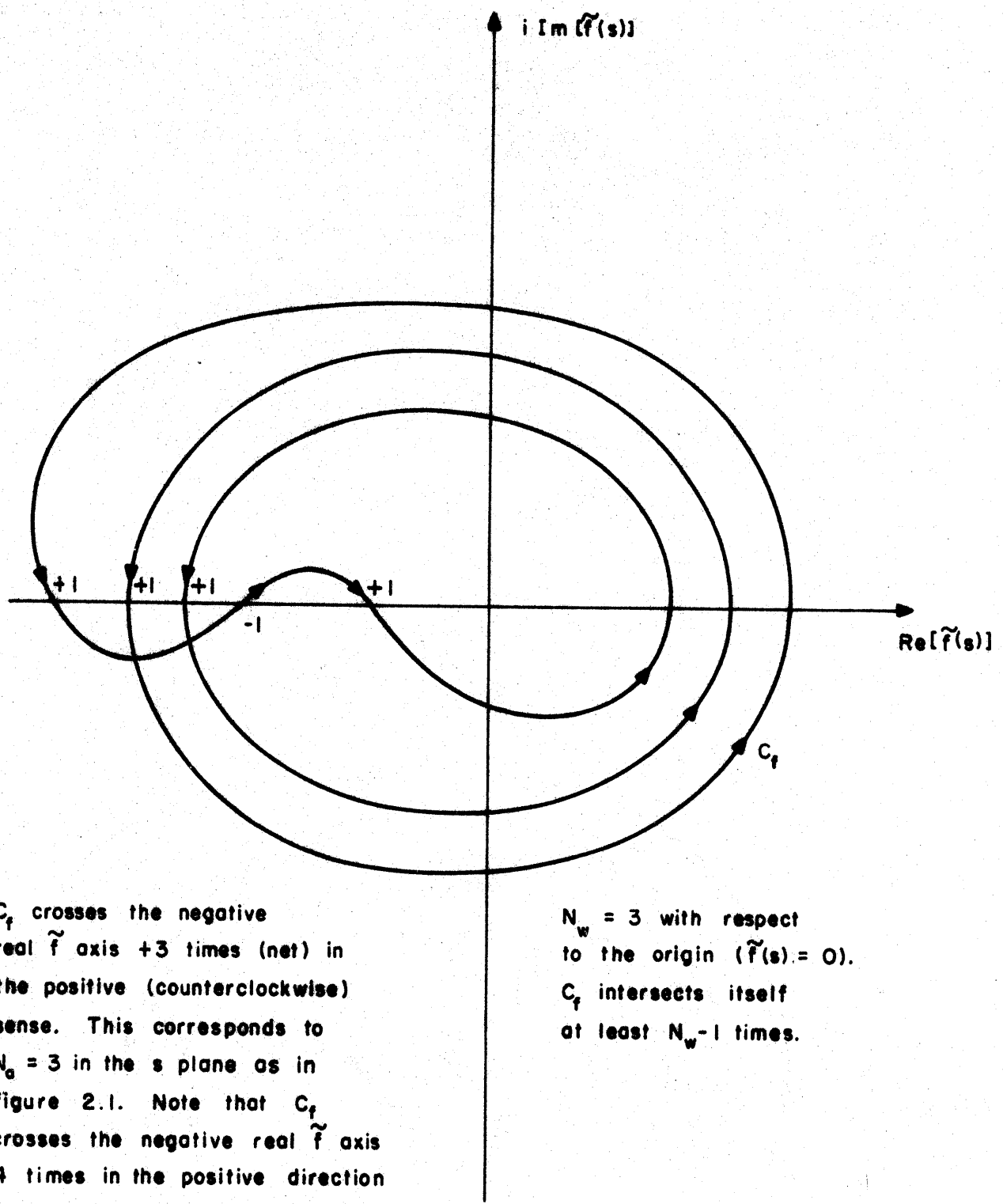
Figure 2.2 Example of Area Subdivision for Localizing Zeros and Poles

argument number to a small radius contour around the zero or pole will quickly determine whether it is a zero or a pole and what its order is.

Thus far we have considered a closed contour C in the s plane. Another interesting way to determine the argument number is to map the contour C in the s plane to an equivalent contour C_f in the $\tilde{f}$ plane where $\tilde{f}(s)$ determines the mapping. As it turns out the argument number N_a of the function $\tilde{f}(s)$ with respect to C in the s plane is equal to the winding number N_w of the contour C_f with respect to the origin ($\tilde{f} = 0$) in the $\tilde{f}$ plane. Remember from the definition of winding number that N_w is the number of positive encirclements (counterclockwise) of the closed contour C_f around the point specified (the origin in this case). In equation form we have

$$N_a\Big|_C = N_w\Big|_{C_f,\tilde{f}=0} \tag{2.16}$$

This result is illustrated in figure 2.3 for the case of $N_w = 3$. In this example C_f is made to correspond to C in the s plane of figure 2.1 in that the crossings of $\arg(\tilde{f}(s))$ through $+\pi$ are preserved, including direction of crossing. Neglecting unspecified details of the function magnitude then figure 2.3 can be considered a mapping of figure 2.1. This contour mapping from s to $\tilde{f}$ plane gives additional insight into the behavior of $\tilde{f}(s)$ since the interior of C is mapped into the "interior" of C_f for zeros only and "exterior" of C_f for poles only where the non simple form of C_f must be included when defining "interior" and "exterior." Note that C_f intersects (crosses) itself at least $N_w - 1$ times. This type of mapping is also used for stability criteria such as the Nyquist criterion.[12]



C_f crosses the negative real $\tilde{f}$ axis +3 times (net) in the positive (counterclockwise) sense. This corresponds to $N_a = 3$ in the s plane as in figure 2.1. Note that C_f crosses the negative real $\tilde{f}$ axis 4 times in the positive direction and once in the negative direction.

$N_w = 3$ with respect to the origin ($\tilde{f}(s) = 0$). C_f intersects itself at least $N_w - 1$ times.

Figure 2.3 Mapping the Contour C from the s Plane to C_f in the $\tilde{f}$ Plane

III. Use of the Argument Number on Conjugate Symmetric Contours

Consider a special type of contour and the argument number that goes with it. Let us call this special type of contour a conjugate symmetric contour. This concept is useful with conjugate symmetric functions which are defined by

$$\overline{\tilde{f}(s)} \equiv \tilde{f}(\bar{s}) \tag{3.1}$$

for all s for which $\tilde{f}(s)$ exists; this same definition applies to vector functions, matrix functions, operators, etc. The complex conjugate is denoted by a bar — over the quantity of concern.

Conjugate symmetric functions are quite important in physical problems due to a property of the Laplace transform (two sided)

$$L[f(t)] \equiv \tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{3.2}$$

with its inverse

$$L^{-1}[\tilde{f}(s)] = f(t) = \frac{1}{2\pi i}\int_{\Omega_o-i\infty}^{\Omega_o+i\infty} \tilde{f}(s)e^{st}ds \tag{3.3}$$

If f(t) is a real valued time function (or vector, matrix, operator, etc.) then $\tilde{f}(s)$ is necessarily conjugate symmetric. Then let us assume that all the functions with which we are dealing such as the dyadic Green's function, its matrix approximation, the excitation and response functions, the general denominator function $\tilde{d}(s)$, etc. have this conjugate symmetry property. There are some quantities such as combined fields, potentials, etc. which do not have this property but do have a generalized form of conjugate symmetry.[10]

As an obvious extension of the concept of a conjugate symmetric function consider a conjugate symmetric contour C as illustrated in figure 3.1. With

$$s \equiv \Omega + i\omega$$
$$\bar{s} = \Omega - i\omega \tag{3.4}$$

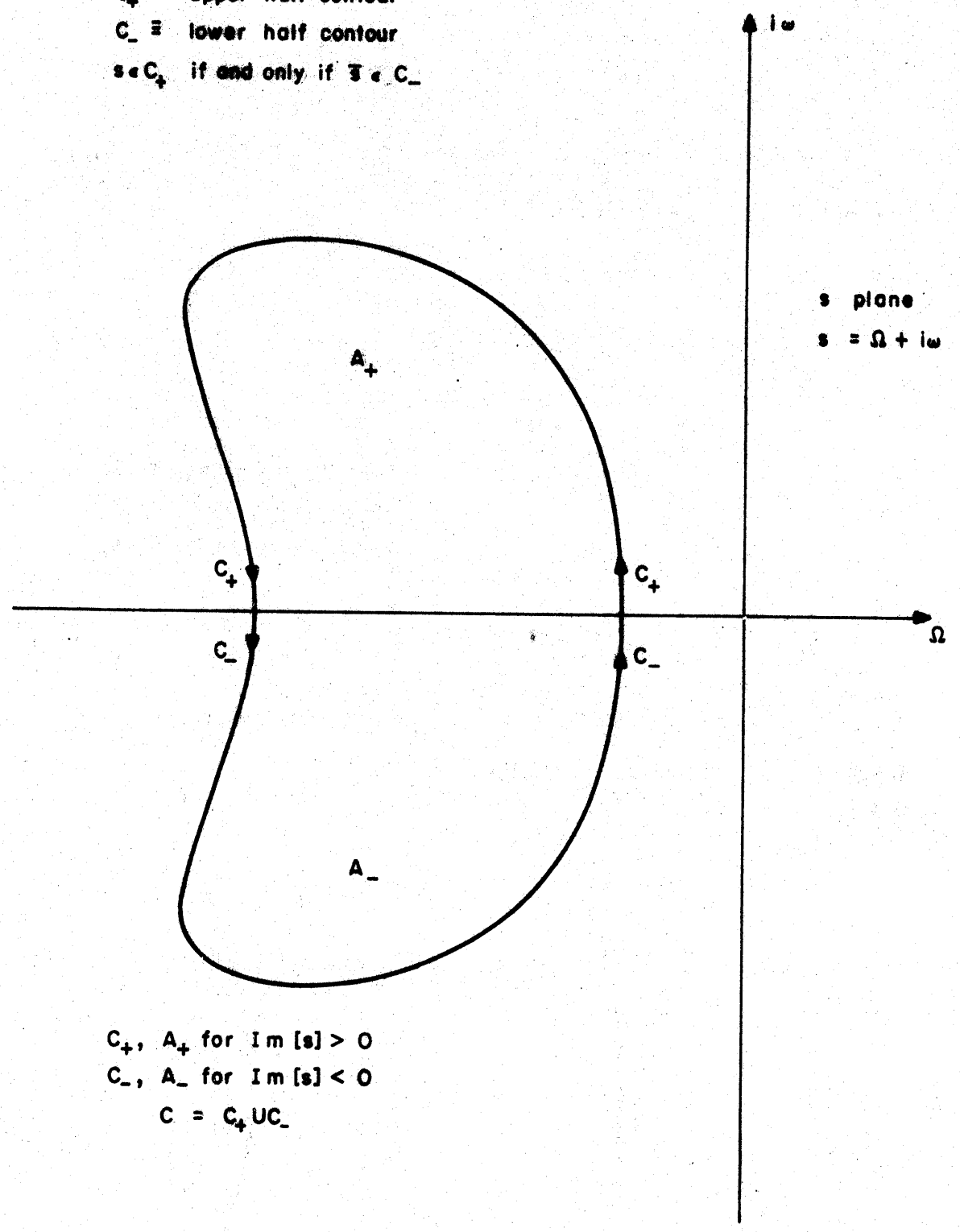


Figure 3.1 Conjugate Symmetric Contour and Half Contours

let C_+ be the portion of C for $\omega > 0$ and C_- be the portion of C for $\omega < 0$. Then C will be called a conjugate symmetric contour provided

$$s \in C_+ \quad \text{if and only if} \quad \bar{s} \in C_- \tag{3.5}$$

C_+ will be called the upper half contour and C_- will be called the lower half contour. Note that C_+ and C_- are both defined in the positive sense such that

$$C = C_+ \cup C_- \tag{3.6}$$

(including limit points on the Ω axis) is a positively directed simple closed contour. Note that if desired one can close C_+ by a contour C_{Ω_+} on the Ω axis; C_- can be closed by a contour C_{Ω_-} on the Ω axis where C_{Ω_+} and C_{Ω_-} are the same contour except that they are oppositely directed. Including C_{Ω_+} and C_{Ω_-} then $C_+ \cup C_{\Omega_+}$ and $C_- \cup C_{\Omega_-}$ are both simple, closed, positively directed contours.

The area A inside C can likewise be split into two areas A_+ for $\omega > 0$ and A_- for $\omega < 0$. Then we have

$$A = A_+ \cup A_- \tag{3.7}$$

including that part of the Ω axis which separates A_+ and A_-.

There is the contour integral identity

$$\oint_C = \int_{C_+} + \int_{C_-} \tag{3.8}$$

Note that we are considering a conjugate symmetric C and a conjugate symmetric function, say $\tilde{f}(s)$. First define

$$F_+ \equiv \int_{C_+} \tilde{f}(s)\,ds$$

$$F_- \equiv \int_{C_-} \tilde{f}(s)\,ds \tag{3.9}$$

Then convert the integral over C_- to one over C_+ using

$$s|_{C_-} = \bar{s}|_{C_+}$$

$$\tilde{f}(s)|_{C_-} = \overline{\tilde{f}(s)}|_{C_+} \tag{3.10}$$

$$ds|_{C_-} = -\overline{ds}|_{C_+}$$

where the $-\overline{ds}$ accounts for the reversal of the end points ("lower" and "upper" limits) in going from C_- to C_+. This gives

$$F_- = \int_{C_-} \tilde{f}(s)\,ds = -\int_{C_+} \overline{\tilde{f}(s)}\;\overline{ds} = -\int_{C_+} \overline{\tilde{f}(s)\,ds} \tag{3.11}$$

leading to

$$\mathrm{Re}[F_-] = -\mathrm{Re}[F_+]$$

$$\mathrm{Im}[F_-] = \mathrm{Im}[F_+] \tag{3.12}$$

$$F_- = -\bar{F}_+$$

Thus a conjugate symmetric contour integral of a conjugate symmetric function $\tilde{f}(s)$ has the general property

$$\oint_C \tilde{f}(s)\,ds = \int_{C_+} \tilde{f}(s)\,ds + \int_{C_-} \tilde{f}(s)\,ds$$

$$= \int_{C_+} \tilde{f}(s)\,ds - \overline{\int_{C_+} \tilde{f}(s)\,ds}$$

$$= 2i\,\mathrm{Im}\left\{\int_{C_+} \tilde{f}(s)\,ds\right\} = 2i\,\mathrm{Im}\left\{\int_{C_-} \tilde{f}(s)\,ds\right\}$$

$$= 2i\int_{C_+} \mathrm{Im}[\tilde{f}(s)\,ds] = 2i\int_{C_-} \mathrm{Im}[\tilde{f}(s)\,ds]$$

$$= 2i\int_{C_+} \{\mathrm{Re}[\tilde{f}(s)]\,d\omega + \mathrm{Im}[\tilde{f}(s)]\,d\Omega\}$$

$$= 2i\int_{C_-} \{\mathrm{Re}[\tilde{f}(s)]\,d\omega + \mathrm{Im}[\tilde{f}(s)]\,d\Omega\} \qquad (3.13)$$

Thus a conjugate symmetric contour integral of a conjugate symmetric function has no real part. Furthermore such a contour integral is found from the imaginary part of a half contour integral, either upper or lower as convenient. Note that the results of equation 3.13 rely only on conjugate symmetry (and integrability), not on analyticity.

Now apply the above results for conjugate symmetric contours to the argument number formulas in section II. Define an upper argument number for $\tilde{f}(s)$ with respect to C_+ as

$$N_{a_+} \equiv \frac{1}{2\pi}\,\mathrm{Im}\left\{\int_{C_+} \frac{1}{\tilde{f}(s')}\,\frac{d}{ds'}\,\tilde{f}(s')\,ds'\right\}$$

$$= \frac{1}{2\pi}\int_{C_+} \mathrm{Im}\,\frac{d\tilde{f}(s')}{\tilde{f}(s')}$$

$$= \frac{1}{2\pi}\,\arg(\tilde{f}(s'))\Big|_{C_+} \qquad (3.14)$$

and a lower argument number as

$$N_{a_-} \equiv \frac{1}{2\pi} \mathrm{Im}\left\{\int_{C_-} \frac{1}{\tilde{f}(s')} \frac{d}{ds'} \tilde{f}(s')ds'\right\}$$

$$= \frac{1}{2\pi} \int_{C_-} \mathrm{Im} \frac{d\tilde{f}(s')}{\tilde{f}(s')}$$

$$= \frac{1}{2\pi} \arg(\tilde{f}(s'))\Big|_{C_+} \qquad (3.15)$$

so that for conjugate symmetric $\tilde{f}(s')$ we have

$$N_{a_+} = N_{a_-}$$

$$N_a = N_{a_+} + N_{a_-} = 2N_{a_+} = 2N_{a_-} \qquad (3.16)$$

Since N_a is an integer, then N_{a+} can be an integer (for N_a even) or an integer + 1/2 (for N_a odd). The same applies to N_{a-}. Due to the symmetry between the upper and lower contours one can concentrate on the upper half contour, and thus N_{a+}, for most considerations.

Figure 3.2 illustrates an example of an argument number calculation for a conjugate symmetric contour. In this example N_a = 3, an odd number. Since a conjugate symmetric function $\tilde{f}(s)$ is real for s real ($s = \Omega$), then $\arg(\tilde{f}(s)) = 0, \pm\pi$ are the only values allowed on the real axis (approached from both sides for $\pm\pi$ case). Figure 3.2 shows the case of 0 argument for most positive Ω (with $\omega = 0$) and $\pm\pi$ argument for the most negative Ω crossing of the real axis by C. However, the situation could be just as easily reversed. If N_a were even in this example then the contour crossings of the Ω axis would have argument 0 in both instances or $\pm\pi$ in both instances.

The even or odd value for N_a has an interesting implication for conjugate symmetric C and $\tilde{f}(s)$. The zeros and poles of $\tilde{f}(s)$ must each come in conjugate symmetric pairs unless they are located on the Ω axis. Odd N_a then necessarily implies that there be at least one zero or pole on the Ω axis inside C. Even N_a might possibly be associated with no zeros or poles on the Ω axis inside C. Then define

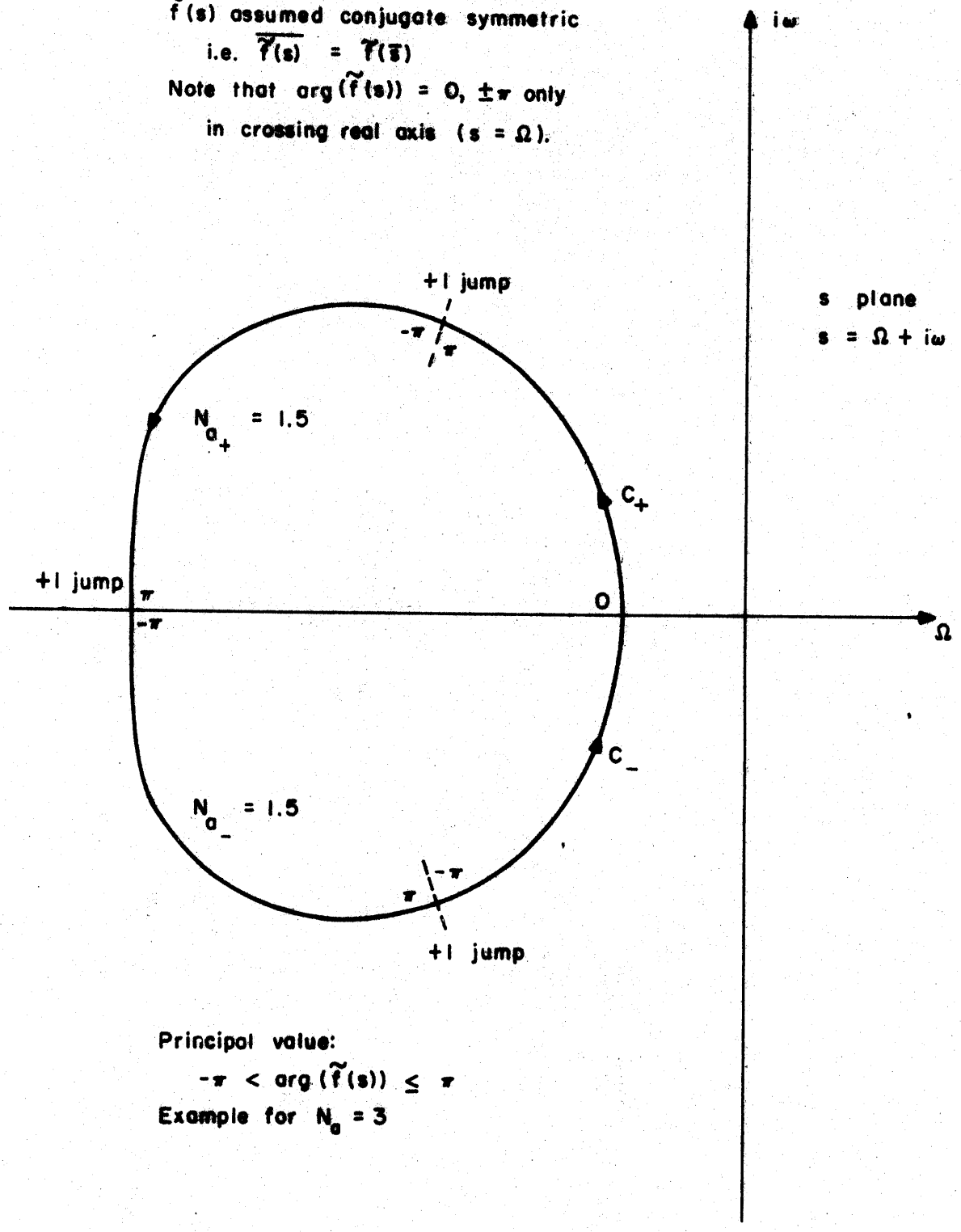


Figure 3.2 Example of Computing the Argument Number from the Principal Values of the Argument on a Conjugate Symmetric Contour

N_{a_Ω} ≡ argument number for Ω axis inside C

$$= N_{o_\Omega} - N_{p_\Omega} \tag{3.17}$$

N_{o_Ω} = number of zeros on Ω axis inside C including multiplicity

N_{p_Ω} = number of poles on Ω axis inside C including multiplicity

Then we have the general result

$$N_a \text{ even} \Leftrightarrow N_{a_\Omega} \text{ even} \Leftrightarrow N_a \text{ integer}$$

$$N_a \text{ odd} \Leftrightarrow N_{a_\Omega} \text{ odd} \Leftrightarrow N_a \text{ integer} + 1/2 \tag{3.18}$$

This concept of a half contour integral is then useful for locating zeros and poles on as well as near the Ω axis.

The use of a half contour integral, say an upper half contour integral, need not explicitly involve the other half to form a symmetric contour. An upper half contour integral need only start on the Ω axis and end at a more negative point on the Ω axis without crossing itself or entering the lower half plane; the associated lower half plane integral is implied. The conjugate symmetry of the function such as $\tilde{f}(s)$ is also implied provided $\tilde{f}(s)$ is real on the Ω axis, since only upper half plane values are used. For computations near the Ω axis then the upper half plane integral can then potentially save computer time by shortening the contour.

Another insight into the argument number is provided by closing the upper half plane contour C_+ with a contour C_{Ω_+} on the Ω axis. Then we can write

$$N_{a_+} = \frac{1}{2\pi i} P \oint_{C_+ \cup C_{\Omega_+}} \frac{1}{\tilde{f}(s')} \frac{d}{ds'} \tilde{f}(s')ds' \tag{3.19}$$

$$= \frac{1}{2\pi} \arg(\tilde{f}(s')) P \oint_{C_+ \cup C_{\Omega_+}}$$

where P indicates the principal value of the contour integral and the corresponding principal value of the argument change around the contour. This is required since $\tilde{f}(s')$ may have one or more zeros or poles on the Ω axis in which case $C_{\Omega+}$ would go right through them. The principal value in this case is πi (instead of $2\pi i$) times $N_{o\Omega} - N_{p\Omega}$ in passing through all the zeros and poles on the Ω axis. The principal value for the $C_{\Omega-}$ integral associated with the lower half contour integral has the same imaginary part even though the integration is in the opposite direction. Note that the principal value is defined so that considering the limits the contour integral value is correct, in this case so that the integral over C is the same as that over $C_+ \cup C_{\Omega+}$ plus that over $C_- \cup C_{\Omega-}$.

Equation 2.9 indicates how to convert the calculation of N_a to numerical form for computer calculation, and equations 2.10 and 2.11 give some specific forms of finite sums for approximating N_a. First choose the discrete s' as s'_m on the conjugate symmetric contour C. In order to take maximum advantage of the conjugate symmetry let us define the s'_m such that they occur in conjugate pairs, except for the one or two of these that lie on the Ω axis. Let s'_o lie on the most positive intersection of C with the Ω axis and require

$$\begin{aligned} &\mathrm{Im}[s'_m] \geq 0 \qquad \text{for } m \geq 0 \\ &\mathrm{Im}[s'_m] \leq 0 \qquad \text{for } m \leq 0 \end{aligned} \tag{3.20}$$

$$\overline{s'_m} = s'_{-m} , \quad \overline{\tilde{f}(s'_m)} = \tilde{f}(s'_{-m})$$

If M is the total number of sample points around C we have the cases

M even

$$\begin{aligned} &\text{2 points on } \Omega \text{ axis} \\ &m = -M', \cdots, M' \text{ (0 included)} \\ &s'_{-M'} = s'_{M'} , \quad M = 2(M' - 1) \end{aligned} \tag{3.21}$$

0 points for Ω axis

$$m = -M', \cdots, M' \quad \text{except } m \neq 0$$
$$s'_{-M'} \neq s'_{M'}, \quad M = 2M' \tag{3.22}$$

M odd

1 point on Ω axis: most positive intersection

$$m = -M', \cdots, M' \ (0 \text{ included})$$
$$s'_{-M'} \neq s'_{M'}, \quad M = 2M' + 1 \tag{3.23}$$

1 point on Ω axis: most negative intersection

$$m = -M', \cdots, M' \quad \text{except } m \neq 0$$
$$s'_{-M'} = s'_{M'}, \quad M = 2M' - 1 \tag{3.24}$$

With these choices of sample point distribution we have approximations for N_a of the form in equations 2.10 and 2.11. Choosing a few selected combinations for M even we have

2 points on Ω axis: equations 2.10 form

$$N_a \simeq \frac{1}{\pi i} \sum_{m=0}^{M'-1} \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)}$$

$$+ \frac{1}{\pi i} \sum_{m=-M'+1}^{0} \frac{\tilde{f}(s'_m) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m) + \tilde{f}(s'_{m-1})}$$

$$= \frac{1}{\pi i} \sum_{m=0}^{M'-1} \left\{ \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)} + \frac{\tilde{f}(s'_{-m}) - \tilde{f}(s'_{-m-1})}{\tilde{f}(s'_{-m}) + \tilde{f}(s'_{-m-1})} \right\}$$

$$= \frac{2}{\pi} \sum_{m=0}^{M'-1} \mathrm{Im} \left[\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)} \right] \tag{3.25}$$

2 points on Ω axis: equations 2.11 form

$$N_a \simeq \frac{1}{4\pi i}\sum_{m=1}^{M'-1}\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}$$

$$+ \frac{1}{4\pi i}\sum_{m=-M'+1}^{-1}\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}$$

$$+ \frac{1}{4\pi i}\frac{\tilde{f}(s'_1) - \tilde{f}(s'_{-1})}{\tilde{f}(s'_o)} + \frac{1}{4\pi i}\frac{\tilde{f}(s'_{-M'+1}) - \tilde{f}(s'_{M'-1})}{\tilde{f}(s'_{M'})}$$

$$= \frac{1}{2\pi}\sum_{m=1}^{M'-1}\mathrm{Im}\left[\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}\right]$$

$$+ \frac{1}{2\pi}\mathrm{Im}\left[\frac{\tilde{f}(s'_1)}{\tilde{f}(s'_o)}\right] - \frac{1}{2\pi}\mathrm{Im}\left[\frac{\tilde{f}(s'_{M'-1})}{\tilde{f}(s'_{M'})}\right] \qquad (3.26)$$

0 points on Ω axis: equations 2.10 form

$$N_a \simeq \frac{1}{\pi i}\sum_{m=1}^{M'-2}\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)}$$

$$+ \frac{1}{\pi i}\sum_{m=-M'+2}^{-1}\frac{\tilde{f}(s'_m) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m) + \tilde{f}(s'_{m-1})}$$

$$+ \frac{1}{\pi i}\frac{\tilde{f}(s'_1) - \tilde{f}(s'_{-1})}{\tilde{f}(s'_1) + \tilde{f}(s'_{-1})} + \frac{1}{\pi i}\frac{\tilde{f}(s'_{-M'}) - \tilde{f}(s'_{M'})}{\tilde{f}(s'_{-M'}) + \tilde{f}(s'_{M'})}$$

$$= \frac{2}{\pi}\sum_{m=1}^{M'-2}\mathrm{Im}\left[\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}\right]$$

$$+ \frac{1}{\pi} \frac{\mathrm{Im}[\tilde{f}(s'_1)]}{\mathrm{Re}[\tilde{f}(s'_1)]} - \frac{1}{\pi} \frac{\mathrm{Im}[\tilde{f}(s'_{M'})]}{\mathrm{Re}[\tilde{f}(s'_M)]} \tag{3.27}$$

0 points on Ω axis: equations 2.11 form

$$N_a \simeq \frac{1}{4\pi i} \sum_{m=2}^{M'-1} \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}$$

$$+ \frac{1}{4\pi i} \sum_{m=-M'+1}^{-2} \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}$$

$$+ \frac{1}{4\pi i} \frac{\tilde{f}(s'_2) - \tilde{f}(s'_{-1})}{\tilde{f}(s'_1)} + \frac{1}{4\pi i} \frac{\tilde{f}(s'_1) - \tilde{f}(s'_{-2})}{\tilde{f}(s'_{-1})}$$

$$+ \frac{1}{4\pi i} \frac{\tilde{f}(s'_{-M'}) - \tilde{f}(s'_{M'-1})}{\tilde{f}(s'_{M'})} + \frac{1}{4\pi i} \frac{\tilde{f}(s'_{-M'-1}) - \tilde{f}(s'_{M'})}{\tilde{f}(s'_{-M'})}$$

$$= \frac{1}{2\pi} \sum_{m=2}^{M'-1} \mathrm{Im}\left[\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_{m-1})}{\tilde{f}(s'_m)}\right]$$

$$+ \frac{1}{2\pi} \mathrm{Im}\left[\frac{\tilde{f}(s'_2) + \overline{\tilde{f}(s'_1)}}{\tilde{f}(s'_1)}\right] - \frac{1}{2\pi} \mathrm{Im}\left[\frac{\overline{\tilde{f}(s'_{M'})} + \tilde{f}(s'_{M'-1})}{\tilde{f}(s'_{M'})}\right]$$

(3.28)

Of these forms for M even, equation 3.25 seems to be the simplest. Which of these is most accurate is not clear without some detailed numerical analysis. A similar set of approximate sums can be formed for M odd, but not perhaps with any significant improvements. All the M odd forms will have extra terms outside the summation with slightly different forms from the terms in the summation. Thus no M odd forms will have as simple a form as equation 3.25 for M even.

Remember that the numerical summations for N_a are only approximate, but are useful in determining whether M (or M' for the upper half contour) has been chosen large enough. By using equation 3.14 to calculate N_{a+} (and thus N_a) from the argument values one can obtain in principle the exact value of N_{a+}. Note that for using the argument calculation corresponding to equation 3.14 it is convenient to choose the s'_m as in equation 3.21 so that the end points are included where $\arg(\tilde{f}(s'))$ is some choice of $0, \pm\pi$ only. This choice conveniently implies M even and the corresponding approximate numerical summations for N_a (or N_{a+}) are found in equations 3.25 and 3.26.

IV. Conservation of Zeros and Poles

An interesting application of the argument number concept is to the conservation of zeros and poles. Alternatively this might be termed a conservation of singularities. This suggests an identification of

$$\text{argument number} \equiv N_a \equiv \text{singularity number} \tag{4.1}$$

One might then think of the conservation of the singularity number. This would seem reminiscent of conservation of charge or conservation of quantum numbers in quantum mechanics.

The concept of the conservation of the singularity number is based on continuous changes of singularity locations in the complex frequency plane as a function of some parameter, P, in a problem of interest. Such a parameter might be an admittance or impedance loading (discrete or continuous) in an antenna or scatterer. Another typical choice could be a ratio of two dimensions in an object of interest. Such parameters could include direction of incidence and polarization (or complementary observation quantities by reciprocity) of some incident electromagnetic (or acoustical, etc.) wave. This parameter list can include just about any parameter one needs to specify in defining an electromagnetic boundary (and initial) value problem.

For present purposes the zeros and poles of interest are considered as continuous functions of some parameter, P. Not all situations meet this requirement. For example, in the continuous deformation of an object (antenna, scatterer) one can have a topologically discontinuous deformation where one part of a surface intersects another part of that surface (or a different surface). The jump from a small "gap" to "zero gap" (or contact) can allow the presence of new natural modes (such as for zero frequency in the near field for an antenna problem) for which the charge is zero, for example. Thus one should avoid discontinuities in the topology or impedance variation as a function of P (our generalized parameter) for the present analysis, or at least account for such discontinuities. Note that our parameter P is not necessarily a scalar (single complex number) quantity, but could be a vector, a matrix, etc.

In earlier sections of this note a general analytic function $\tilde{f}(s)$ is introduced; it has zeros and/or poles in portions of the s plane of interest. A general denominator function $\tilde{d}(s)$ with only zeros in portions of the s plane (or entire s plane) is also considered. Let such functions have poles and zeros of only finite order in the region of the s plane of interest, i.e., allow no essential singularities in the finite s plane of interest (for all values of P of interest). For the

singularity expansion of the electromagnetic response of finite size simple objects (including well behaved media) in free space there is known to be no such essential singularity in the finite s plane.[5,6]

Considering our general function $\tilde{f}(s)$ it can be strictly considered as a function of both s and P as $\tilde{f}(s,P)$. If for all s in some neighborhood of a finite order zero of $\tilde{f}(s)$ (or of $\tilde{f}^{-1}(s)$), $\tilde{f}(s)$ (or $\tilde{f}^{-1}(s)$) is a continuous function of P, then the zero (or pole) is a continuous function of P. A common form for a zero or pole of $\tilde{f}(s,P)$ would be

$$\tilde{g}(s,P) = \tilde{f}(s,P) = 0 \qquad \text{for zeros} \tag{4.2}$$

or

$$\tilde{g}(s,P) = \tilde{f}^{-1}(s,P) = 0 \qquad \text{for poles} \tag{4.3}$$

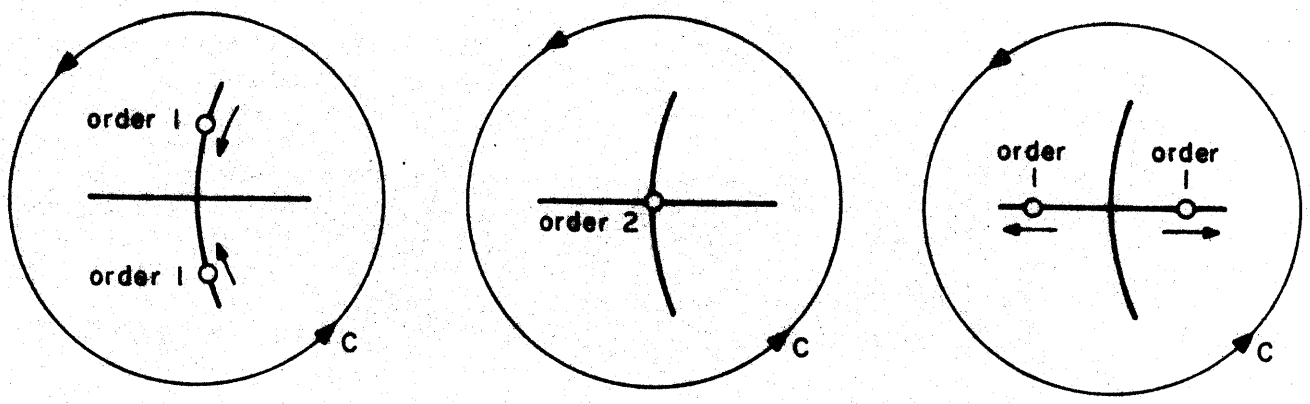
which defines a zero or pole s as a function of P. In a simple form one might have

$$\tilde{g}(s,P) = \tilde{g}(s) + P = 0 \tag{4.4}$$

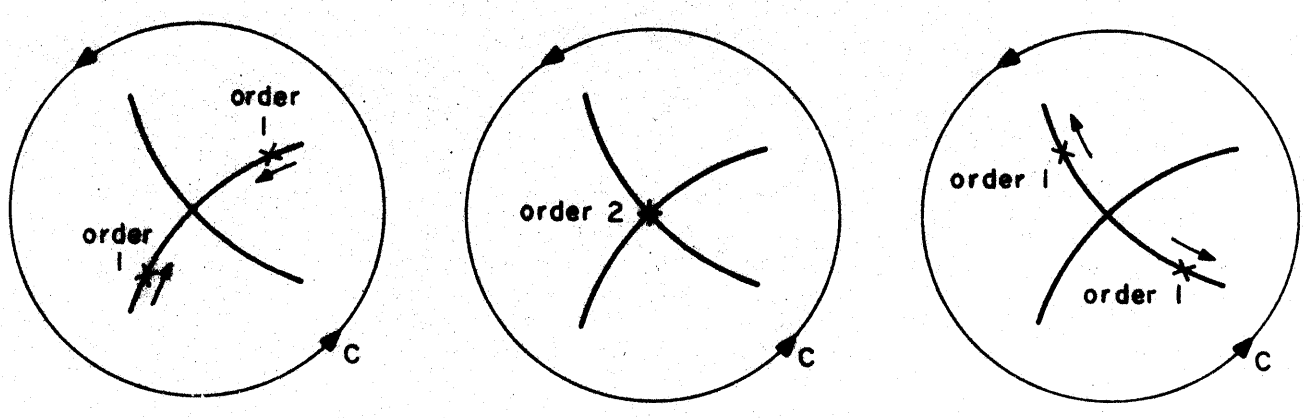
This is a form commonly encountered in root locus studies such as used in circuit theory (or theory of some other types of linear systems).[13] In the simpler form of equation 4.4, or the more general forms in equations 4.2 and 4.3, root locus procedures are useful for natural frequency considerations for the singularity expansion method.

Generalizing the above concept define a simple closed positive contour C in the s plane as indicated in figure 4.1. Let there be one or more zeros or poles of $\tilde{f}(s)$ inside C and bounded away from the contour for the variation of P over some small range being considered. Then $\tilde{f}(s)$ is a continuous function of P on the contour C and bounded away from zero on the contour C. The argument number N_a of $\tilde{f}(s)$ with respect to the contour C can be calculated as discussed in section II. Since the integrand and $\arg(\tilde{f}(s))$ are continuous functions of P for s on C then N_a is a continuous function of P. Then since N_a is an integer N_a must be a constant, i.e., not a function of P. This is what is meant by conservation of argument number or conservation of singularity number.

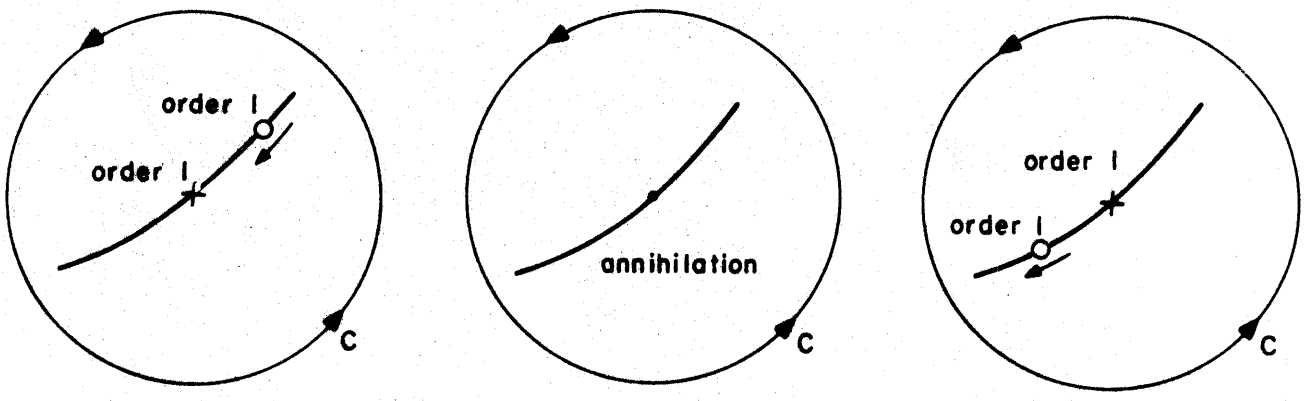
Figure 4.1 gives some examples of types of behavior consistent with the conservation of singularity number. Consider two first order zeros near each other in the s plane and inside C as illustrated in figure 4.1A. As the parameter P is varied



A. Two simple zeros coalescing and splitting: $N_a = 2$



B. Two simple poles coalescing and splitting: $N_a = -2$



C. Simple zero passing through simple pole: $N_a = 0$

Figure 4.1 Some Examples of Conservation of Singularity Number as a Parameter is Varied

one may observe the two zeros coalescing and splitting to form two zeros again. The singularity number of each zero is +1 and of all those inside C is +2. Therefore at the coalescing point the zero must be second order and after the splitting the zeros are each first order again. Note that over the range of variation of P of interest no singularities are allowed to cross C. In such a circumstance C must be redefined such that no such crossing occurs.

An example of this coalescing of two zeros to form a second order zero would be a denominator function $\tilde{d}(s)$ such as a determinant which has only zeros in the portion of the s plane of interest. Such an example occurs in a recent note by Tesche where he considers radiation from a resistively loaded radiating antenna.[2] He observes for each of several loading functions (resistive) that as the coefficient of the resistance function is increased through positive real values a critical point occurs on the negative Re[s] axis. At this point the lowest order conjugate pair of natural frequencies coalesces together on the Re[s] axis and then splits to go in opposite directions on the Re[s] axis. The conservation of the singularity number implies that such a zero at the coalescing point be second order. If the numerator function has no zero there then this would imply the existence of a second order pole in the response (current, field, etc.) of such a structure. Such an antenna or scatterer might be referred to as critically damped.

Figure 4.1B shows a case with singularity number -2, with two first order poles coalescing. This case is similar to the two zero case discussed above. Just consider $1/\tilde{d}(s)$ as our function of interest and all the properties of zero trajectories of $\tilde{d}(s)$ carry over to pole trajectories.

Figure 4.1C shows a first order zero and a first order pole coalescing and splitting. The singularity number of the zero is +1 and of the pole is -1 giving a singularity number of zero for the process described. At the point of coalescing the singularities can be thought of as annihilating each other in that there are no singularities inside C.

An example of such pole-zero annihilation occurs in the scattering of electromagnetic waves by symmetric objects. For objects with a symmetry plane or symmetry axis the fields and currents decompose according to this symmetry.[4] In particular the natural modes for the current $\underset{\sim}{\nu}_\alpha$ and the coupling vectors $\underset{\sim}{\mu}_\alpha$ exhibit this symmetry. For certain angles of incidence and polarization of the incident wave there is no similar symmetry component in the incident wave and the coupling coefficient (equation 1.3) must be zero. If the coupling coefficient is zero a pole (or more commonly a pole pair with conjugate symmetry) is no longer present in the response. Thus as the angle of incidence and/or polarization are continuously varied to the

case giving a zero coupling coefficient a zero in the response must move to the pole of interest to annihilate it and thereby conserve singularity number.

Since an electromagnetic response (the numerator function) is typically a vector valued function of position then the zero is not necessarily at the same frequency for each vector component, except of course at the point of coalescing (the natural frequency). Such vector functions can be scalarized by use of dot products or symmetric products involving integrals over the body. This would make the response zero occur at a single frequency which varies with the parameter. An example of such a numerator zero would be the zeros of an admittance function (corresponding to the poles of the impedance). Thus the concept of singularity conservation leads, at least in certain cases, necessarily to the existence of zeros in the electromagnetic response (current densities, scattered fields, etc.). This is clearly an important avenue of research in electromagnetic theory. There is a considerable amount known about expanding solutions in terms of poles. One might be able to use the zeros as well.

Seeing that the concept of conservation of singularities gives some interesting insight into certain properties of the electromagnetic response of objects one might try to generalize the concept somewhat. First one might allow branch points inside C, provided there were more than one, and they were of such a form that the associated branch cut(s) could be kept from crossing C. Then the function $\tilde{f}(s)$ could still be analytic and non zero on C giving an integer value for N_a. Thus the concept of singularity number could readily include certain restricted types of branches.

One might go even further and define some kind of generalized argument number or singularity number which allows for branch cuts to cross C, or perhaps lets the contour integral be over a contour C which wanders over more than one Riemann sheet of the function $\tilde{f}(s)$ which is multivalued. Such types of argument number can be defined in forms similar to those in equation 2.5. However, it is not clear to me at this writing what the best form of a generalized singularity number would be or what practical importance it might have. I leave this with the reader as food for thought.

V. Finding Zeros Inside the Contour from Function Values on the Contour

As discussed earlier (section I) our function of interest may be lengthy to calculate from its definition (say as a determinant). For this reason a set of M points s'_m was defined in counterclockwise order around a simple closed contour C and the function values calculated at these M points. Section II used these M points to find the argument number N_a, including summations over these points to approximate a contour integral.

This section considers finding zeros inside C from the M function values on C. This will give a way of finding approximate values of the zeros of the function inside C provided it has no poles inside C (or the poles are first removed).

Let the function $\tilde{f}(s)$ be analytic everywhere inside and on C where C is a contour as illustrated in figure 1.1. There is the well known Cauchy integral theorem[11,14,15]

$$0 = \oint_C \tilde{f}(s')ds' \tag{5.1}$$

which gives a relation involving only the values of the function on the contour. A slightly more complicated expression known as the Cauchy integral formula is

$$\tilde{f}(s) = \frac{1}{2\pi i} \oint_C \frac{\tilde{f}(s')}{s' - s} ds' , \qquad s \in A \tag{5.2}$$

where A is the part of the s plane inside C. This result is readily generalized to nth order derivatives where n is a positive integer as

$$\frac{d^n}{ds^n} \tilde{f}(s) = \frac{n!}{2\pi i} \oint_C \frac{\tilde{f}(s')}{(s' - s)^{n+1}} ds' , \qquad s \in A \tag{5.3}$$

$$n = 0, 1, 2, 3, \cdots$$

As was done with the argument number integral discussed in section II (equation 2.5) this contour integral (equation 5.3) can be approximated as a sum which we can write as

$$\frac{d^n}{ds^n}\tilde{f}(s) \simeq \frac{n!}{2\pi i}\oint_C \frac{\tilde{f}(s')\Delta s'}{(s' - s)^{n+1}} \tag{5.4}$$

Now s' lies on the contour and assumes the values s'_m for m = 1, ···, M in ascending order around the contour in the positive (counterclockwise) sense. The various possible approximate contour sums involve the choice of the s'_m and the manner in which $\tilde{f}(s')$, $(s - s')^{n+1}$, and $\Delta s'$ are defined in terms of the nearby values of s'_m and the associated function values $\tilde{f}(s'_m)$. One choice is

$$\Delta s' = s'_{m+1} - s_m$$

$$s' \simeq \frac{1}{2}[s'_{m+1} + s'_m]$$

$$\tilde{f}(s') \simeq \frac{1}{2}[\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)] \tag{5.5}$$

$$\frac{d^n}{ds^n}\tilde{f}(s) \simeq \frac{n!}{4\pi i}\sum_{m=1}^{M} \frac{\tilde{f}(s'_{m+1}) + \tilde{f}(s'_m)}{\left[\frac{1}{2}(s'_{m+1} + s'_m) - s\right]^{n+1}}[s'_{m+1} - s'_m]$$

$$s'_{M+1} \equiv s'_1$$

A second choice is

$$\Delta s' = \frac{1}{2}[s'_{m+1} - s'_{m-1}]$$

$$s' = s'_m$$

$$\tilde{f}(s') = \tilde{f}(s'_m)$$

$$\frac{d^n}{ds^n}\tilde{f}(s) \simeq \frac{n!}{4\pi i}\sum_{m=1}^{M} \frac{\tilde{f}(s'_m)}{[s'_m - s]^{n+1}}[s'_{m+1} - s'_{m-1}]$$

$$s'_{M+1} \equiv s'_1 \qquad (5.6)$$

$$s'_0 \equiv s'_M$$

The first choice bases the terms on averages between the values at s'_m and s'_{m+1}. The second choice bases the terms on their values at s_m, except for $\Delta s'$ which uses values from both s'_{m+1} and s'_{m-1}.

Now that we have approximations for the function and its derivatives inside the contour in terms of the M function values on the contour these formulas can be used for various purposes. One use for these formulas is to find approximate values of the zeros of $\tilde{f}(s)$ inside C. There are various ways to find the zeros of analytic functions of a complex variable, such as Newton's method and Muller's method.[9] Equations 5.5 and 5.6 give ways to rapidly evaluate the function values and its derivatives (for values of s inside C) which are needed in such iterative zero searching procedures. One should be careful that the values of s used in the iterations do not wander too close to C or outside of C because one expects the procedure to break down in such cases.

Having found an approximate value s_α for a zero of $\tilde{f}(s)$ inside C then one can replace each of the function values on the contour by dividing by $s'_m - s_\alpha$, i.e., replace

$$\tilde{f}(s'_m) \rightarrow \tilde{f}(s'_m)/(s'_m - s_\alpha) \qquad (5.7)$$

Then the new approximate values of the function inside C will have a zero, s_α, removed (at least approximately removed). The modified function values on the contour (from equation 5.7) can then be used in equations 5.5 and/or 5.6 to generate function values and derivatives needed in finding another zero. This process goes on until all the zeros inside C are found. Note that from the argument number around C as discussed in section II one knows beforehand how many zeros inside C are to be found. Multiple order zeros will be found as two or more values of s_α found close together by the above procedure. Note that by hypothesis f(s) has no poles inside C to confuse a zero count from the argument number around C.

An alternative procedure for finding zeros inside C is to follow lines of constant $\arg(\tilde{f}(s))$ from C inward until a zero is reached or the path returns to C. Equations 5.5 and 5.6 can be used to approximate $\arg(\tilde{f}(s))$. All constant $\arg(\tilde{f}(s))$ paths leaving a zero of $\tilde{f}(s)$ must cross C since there are no poles inside C by hypothesis. Thus constant $\arg(\tilde{f}(s))$ lines exist from C to every zero inside C. This fact can be used to reduce

the search for zeros over A (the area inside C) to a search over C in some sense. Again the value of N_a around C tells when all the zeros are found. If there is a question of the order of a particular zero found by this procedure then the argument number can be determined on a small radius contour around the zero in question.

Yet another way to use equations 5.5 and 5.6 for finding the zeros of $\tilde{f}(s)$ inside C is by area subdivision of A, giving several smaller contours as discussed in section II. In order to use this technique to locate the aeros one needs the values of $\arg(\tilde{f}(s))$ on the smaller contours so as to evaluate the argument number for each contour. Equations 5.5 and 5.6 can be used to provide approximate values of $\arg(\tilde{f}(s))$ for portions of the smaller contours not on C. Using this technique the procedure of area subdivision need not require calculation of additional values of $\tilde{f}(s)$ from its original definition (which by hypothesis requires relatively lengthy computation).

In order to understand the pole-zero pattern of a complex function such as a determinant, a technique which has been used is to plot contours of constant $|\tilde{f}(s)|$ in the s plane.[3,8] To do so requires that one be able to calculate $\tilde{f}(s)$ and equations 5.5 and 5.6 do this without resorting to the original definition of $\tilde{f}(s)$. Thus for contour plotting purposes the contour sums on C provide a means of calculating contours of the function values inside (but not too near) C. As will be discussed later contours of constant $\ln(|\tilde{f}(s)|)$ and constant $\arg(\tilde{f}(s))$ with uniform increments are both interesting. This technique of generating function values inside C from those on C could prove useful for both argument and logarithmic magnitude contour plots. In a sense this is a combination of the Cauchy integral formula with the argument number formula into one complex variable formula.

In section III the concept of a conjugate symmetric contour was defined as well as a conjugate symmetric distribution of the sample points s_m on C. Equations 3.21 through 3.24 give four ways to arrange the M points s'_m, depending on whether M is even or odd, preserving conjugate symmetry. Consider the forms of the sums in equations 5.5 and 5.6 and pair terms in the sums according to conjugate symmetric s'_m and thus also conjugate symmetric $\tilde{f}(s'_m)$. The sums themselves are thus conjugate symmetric with respect to s. This procedure assures that the approximation to $\tilde{f}(s)$ is itself conjugate symmetric provided the original function values used to generate the sums are themselves conjugate symmetric.

In the case of the argument number the conjugate symmetric contour is easily reduced to a half contour and the corresponding approximating sums are reduced by about half. However, the Cauchy integral formula and related expressions (equation 5.3) involve two complex variables, s' and s. This prevents the

reduction in the number of terms in the sum unless s is real, in which case the upper half contour can be used and the real part taken with proper allowance for whether individual terms are counted once or twice depending on their location relative to the real axis. Of course only about half the $\tilde{f}(s'_m)$ values need to be calculated from the original definition of $\tilde{f}(s)$ since the remaining values can be found by complex conjugation. It is then in the fact that conjugate symmetric s'_m and $\tilde{f}(s'_m)$ are used that some computation benefits can be obtained.

VI. Saddles

An interesting type of point in the complex plane might be called a saddle. By a saddle is meant a point in the s plane which we denote by s_Σ such that

$$\tilde{f}(s) \text{ is analytic at } s = s_\Sigma$$

$$\tilde{f}(s_\Sigma) \neq 0 \tag{6.1}$$

$$\frac{d}{ds}\tilde{f}(s)\Big|_{s=s_\Sigma} = 0$$

There are many such points in the s plane and thus Σ is some appropriate index set to label the saddles. An nth order saddle is defined by

$$\frac{d^m}{ds^m}\tilde{f}(s)\Big|_{s=s_\Sigma} = 0 \qquad \text{for } m = 1, 2, \cdots, n$$

$$\frac{d^{n+1}}{ds^{n+1}}\tilde{f}(s)\Big|_{s=s_\Sigma} \neq 0 \tag{6.2}$$

$$\tilde{f}(s_\Sigma) \neq 0$$

$$\tilde{f}(s) \text{ analytic at } s = s_\Sigma$$

Saddles are important points in the s plane for constructing contour plots for $\tilde{f}(s)$ and for root locus procedures if the parameter P for variation is combined with $\tilde{f}(s)$ in a simple form such as the standard root locus equation[13]

$$\tilde{f}(s) + P = 0 \tag{6.3}$$

Here P might be a real variable such as a resistance or conductance or a complex function of s such as an impedance or admittance. Alternatively our root locus equation may take the more general form

$$\tilde{f}(s,P) = 0 \tag{6.4}$$

Additional considerations involving the variation of the roots as P is varied are discussed in section IV.

One of the properties of saddles is that they are insensitive to some degree of the manner of defining the function $\tilde{f}(s)$. To illustrate consider a function $\tilde{F}(\tilde{f}(s))$, assumed analytic at $\tilde{f}(s_\Sigma)$. Note that

$$\frac{d}{ds}\tilde{F}(\tilde{f}(s)) = \frac{d}{d\tilde{f}}\tilde{F}\,\frac{d\tilde{f}(s)}{ds} = 0 \tag{6.5}$$

implies

$$\frac{d}{ds}\tilde{f}(s) = 0 \tag{6.6}$$

provided

$$\frac{d}{d\tilde{f}}\tilde{F}(\tilde{f}) \neq 0 \tag{6.7}$$

Thus the saddle location is in general preserved under such a transformation. Similarly considering multiple order saddles suppose

$$\frac{d^m}{ds^m}\tilde{F}(\tilde{f}(s)) = 0 \qquad \text{for } m = 1, 2, \cdots, n$$

$$\frac{d^{n+1}}{ds^{n+1}}\tilde{F}(\tilde{f}(s)) \neq 0 \tag{6.8}$$

$$\frac{d}{d\tilde{f}}\tilde{F}(\tilde{f}) \neq 0$$

Then assuming $\tilde{F}$ is an analytic function of $\tilde{f}$ at $s = s_\Sigma$ (or $\tilde{f}(s) = \tilde{f}(s_\Sigma)$) a Taylor series representation of $\tilde{F}$ is possible in terms of $\tilde{f}$ with leading term as

$$\tilde{F}(\tilde{f}(s)) = F_o + O(\tilde{f}) , \qquad F_o \neq 0 \tag{6.9}$$

where the leading term is a constant, consistent with the last of equations 6.8. Expanding both $\tilde{F}$ and $\tilde{f}$ near s_Σ in Taylor series gives leading terms as

$$\tilde{F}(\tilde{f}(s)) = F_o + F'_{n+1}(s - s_\Sigma)^{n+1} + O\left((s - s_\Sigma)^{n+2}\right), \quad F_o \neq 0$$

$$\tilde{f}(s) = f_o + f_{q+1}(s - s_\Sigma)^{q+1} + O\left((s - s_\Sigma)^{q+2}\right), \quad f_q \neq 0 \tag{6.10}$$

Equations 6.8 and 6.10 require that

$$q = n \tag{6.11}$$

Equations 6.2 and 6.10 require the same. Thus the sets of requirements in equations 6.2 and 6.8 are equivalent.

An example of the above discussed insensitivity of the saddles s_Σ to transformations of the function $\tilde{f}(s)$ (or effectively to the definition of $\tilde{f}(s)$ in some degree) can be illustrated by an example. If $\tilde{f}(s)$ has saddles s_Σ then $1/\tilde{f}(s)$ has the same saddles and conversely. If $\tilde{f}(s)$ is a driving point admittance then $1/\tilde{f}(s)$ is a driving point impedance. Hence driving point admittances and driving point impedances have the same saddles.

Consider an nth order saddle s_Σ. Near the saddle the function has the form

$$\tilde{f}(s) = f_o + f_{n+1}(s - s_\Sigma)^{n+1} + O\left((s - s_\Sigma)^{n+2}\right)$$

$$\tilde{f}(s_\Sigma) = f_o \neq 0 \tag{6.12}$$

$$f_{n+1} \neq 0$$

Construct contours in the s plane for

$$|\tilde{f}(s)| = |f_o|$$

$$\arg(\tilde{f}(s)) = \arg(f_o) \tag{6.13}$$

The non zero term $f_{n+1}(s - s_\Sigma)^{n+1}$ in the expansion makes the contour pattern from equations 6.13 near s_Σ have $2(n + 1)$ magnitude contours ($|\tilde{f}(s)| = |f_o|$) and $2(n + 1)$ argument contours ($\arg(\tilde{f}(s)) = \arg(f_o)$), all radiating from s_Σ with the magnitude and argument contours alternating around s_Σ and separated from each other by an angle $2\pi/[4(n + 1)]$.

One important use for saddles is their role as critical points in contour plots of $\tilde{f}(s)$ in the s plane. In a contour plotting procedure to be discussed in the next section $\ln(|\tilde{f}(s_\Sigma)|)$ and $\arg(\tilde{f}(s_\Sigma))$ are critical values for contour plots of $\ln(|\tilde{f}(s)|)$ and $\arg(\tilde{f}(s))$ in the s plane. In particular $\arg(\tilde{f}(s_\Sigma))$ is a special value of $\arg(\tilde{f}(s))$ which gives contours which divide the constant $\arg(\tilde{f}(s))$ contours emanating from one zero from those emanating from another zero. They similarly divide the constant argument contours associated with one pole from those associated with another. This characteristic of $\arg(\tilde{f}(s)) = \arg(\tilde{f}(s_\Sigma))$ lines also divides constant argument lines according to poles. Viewed another way constant argument lines go from zeros to poles (including zeros or poles at infinity) and each set of argument curves defined by a zero-pole combination is separated in the s plane from other such sets by $\arg(\tilde{f}(s_\Sigma))$ lines. Constant $\arg(\tilde{f}(s))$ contours only intersect at zeros, poles, and saddles (provided $\tilde{f}(s)$ is analytic elsewhere).

Saddles can be found by use of the argument number integral. The argument number depends on both the function and the contour. Thus the argument number in sections II, III, and IV which is discussed in terms of the function $\tilde{f}(s)$ and the contour C can be denoted by $N_a(\tilde{f}(s),C)$. Since a saddle is defined from equations 6.1 and 6.2 by the requirement that $\tilde{f} \neq 0$ while $d\tilde{f}/ds = 0$ then saddles can be found as zeros of $d\tilde{f}/ds$ with the restriction that second and higher order zeros of $\tilde{f}$ are excluded. Thus if a simple closed contour C_Σ with positive direction is defined in the s plane, and if C_Σ encloses no zeros of second or higher order and no poles of $\tilde{f}(s)$, then we have

$$N_{a_\Sigma} \equiv N_a\left(\frac{d}{ds}\,\tilde{f}(s),\ C_\Sigma\right)$$

$$= \frac{1}{2\pi i}\oint_{C_\Sigma}\left[\frac{d}{ds'}\,\tilde{f}(s')\right]^{-1}\frac{d^2}{ds'^2}\,\tilde{f}(s')ds'$$

$$= \frac{1}{2\pi i}\oint_{C_\Sigma}\left[\frac{d}{ds'}\,\tilde{f}(s')\right]^{-1} d\left[\frac{d}{ds'}\,\tilde{f}(s')\right]$$

$$= \frac{1}{2\pi i} \ell n\left[\frac{d}{ds'} \tilde{f}(s')\right] \Big|_{C_\Sigma}$$

$$= \frac{1}{2\pi} \arg\left[\frac{d}{ds'} \tilde{f}(s')\right] \Big|_{C_\Sigma}$$

$$= N_\Sigma$$

$\equiv$ saddle number

= number of saddles inside C_Σ including multiplicity

$$= \sum_{A_\Sigma} n_{\Sigma_j} \tag{6.14}$$

where

A_Σ is the open region enclosed by C_Σ and

n_{Σ_j} = multiplicity of jth saddle in A_Σ (6.15)

The considerations in section II concerning the argument number integral can be applied to the saddle number considered here. The use of conjugate symmetric contours and associated half contours also carries over from section III. The function $\tilde{f}$ has been replaced, however, by df/ds.

For numerical approximations to the saddle number N_Σ one can first consider approximating $d\tilde{f}/ds$ on the contour by

$$\frac{d}{ds'} \tilde{f}(s') \simeq \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{s'_{m+1} - s'_m} \equiv f'_m \tag{6.16}$$

where this approximation is appropriate for some value of s' between s'_m and s'_{m+1} on the contour C_Σ. As before m = 1, 2, $\cdots$, M denote a sequence of M points in a positive order around C_Σ. As long as

$$|\arg(f'_{m+1}) - \arg(f'_m)| = |\Delta_m[\arg(f'_m)]| << \pi \qquad (6.17)$$

as m progresses around the contour, then the change in $\arg(f'_m)$ around the contour is $2\pi N_\Sigma$. The saddle number so calculated is exactly an integer since the principal value of $\arg(f'_m)$ must return to its starting value on going around the contour. Provided that Δ_m as in equation 6.17 is small for all m then the saddle number calculation is exact. Note that the f'_m are defined in terms of the $\tilde{f}(s'_m)$ so that derivatives of $\tilde{f}(s')$ need not be calculated.

In a manner similar to that in section II approximate numerical integral formulas for N_Σ can be developed. Let us use

$$N_\Sigma = \frac{1}{2\pi i}\oint_{C_\Sigma}\left[\frac{d}{ds'}\tilde{f}(s')\right]^{-1} d\left[\frac{d}{ds'}\tilde{f}(s')\right]$$

$$\simeq \frac{1}{2\pi i}\sum_{C_\Sigma}\left[\frac{d}{ds'}\tilde{f}(s')\right]^{-1}\Delta\left[\frac{d}{ds'}\tilde{f}(s')\right] \qquad (6.18)$$

Approximate the terms in this formula in a symmetric sense centered about s'_m starting from equation 6.16 as

$$\Delta\left[\frac{d}{ds'}\tilde{f}(s')\right] \simeq \frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{s'_{m+1} - s'_m} - \frac{\tilde{f}(s'_m) - \tilde{f}(s'_{m-1})}{s'_m - s'_{m-1}}$$

$$= \frac{(s'_m - s'_{m-1})\tilde{f}(s'_{m+1}) + (-s'_{m+1} + s'_{m-1})\tilde{f}(s'_m) + (s'_{m+1} - s'_m)\tilde{f}(s'_{m-1})}{(s'_{m+1} - s'_m)(s'_m - s'_{m-1})} \qquad (6.19)$$

$$\frac{d}{ds'}\tilde{f}(s') \simeq \frac{1}{2}\left\{\frac{\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)}{s'_{m+1} - s'_m} + \frac{\tilde{f}(s'_m) - \tilde{f}(s'_{m-1})}{s'_m - s'_{m-1}}\right\}$$

$$= \frac{1}{2}\,\frac{(s'_m - s'_{m-1})\tilde{f}(s'_{m+1}) + (s'_{m+1} - 2s'_m + s'_{m-1})\tilde{f}(s'_m) + (-s'_{m+1} + s'_m)\tilde{f}(s'_{m-1})}{(s'_{m+1} - s'_m)(s'_m - s'_{m-1})}$$

The approximating sum is then

$$N_\Sigma \simeq \frac{1}{\pi i}\sum_{m=1}^{M}\frac{(s'_m - s'_{m-1})(\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)) - (s'_{m+1} - s'_m)(\tilde{f}(s'_m) - \tilde{f}(s'_{m-1}))}{(s'_m - s'_{m-1})(\tilde{f}(s'_{m+1}) - \tilde{f}(s'_m)) + (s'_{m+1} - s'_m)(\tilde{f}(s'_m) - \tilde{f}(s'_{m-1}))}$$

$$= \frac{1}{\pi i}\sum_{m=1}^{M}\frac{(s'_m - s'_{m-1})\tilde{f}(s'_{m+1}) + (-s'_{m+1} + s'_{m-1})\tilde{f}(s'_m) + (s'_{m+1} - s'_m)\tilde{f}(s'_{m-1})}{(s'_m - s'_{m-1})\tilde{f}(s'_{m+1}) + (s'_{m+1} - 2s'_m + s'_{m-1})\tilde{f}(s'_m) + (-s'_{m+1} + s'_m)\tilde{f}(s'_{m-1})} \qquad (6.20)$$

$$s'_{M+1} \equiv s'_1$$

$$s'_0 \equiv s'_M$$

Again the numerical formulas involve only the s'_m and the $\tilde{f}(s'_m)$. If these have been calculated for other purposes, such as on the contour C, on a portion of the contour C, or derived from the values on C by the Cauchy integral formula, then the use of equation 6.20 can be numerically efficient.

The above formulas can be applied to conjugate symmetric contours and the associated half contours as discussed in section III. Some saving in the number of points s'_m and function values $\tilde{f}(s'_m)$ (approximately a factor of two saving) to be calculated from the original definition is obtained due to the conjugate symmetry.

An important property of saddles is that under certain restrictions they are conserved in a sense similar to poles and zeros as discussed in section IV. Stated another way the saddle number N_Σ, like the singularity number N_a, is conserved as some parameter P is varied.

If one considers a generalized saddle number as being precisely the argument number of $d\tilde{f}(s)/ds$ then the generalized saddle number N_Σ is conserved just like the argument number N_a as discussed in section IV. However, the generalized saddle number will include contributions from poles of $\tilde{f}(s)$ and second or higher order zeros of $\tilde{f}(s)$. This more general form can then be referred to as conservation of generalized saddles.

If our attention is restricted to cases for which C_Σ includes no poles of $\tilde{f}(s)$ and no second or higher order zeros of $\tilde{f}(s)$, then the saddle number applies to conservation of saddles in the strict sense. For such cases the coalescing and splitting of two zeros or two poles as in figure 4.1 illustrates the coalescing and splitting of two saddles as well. More general cases involving several saddles of various orders are also possible.

VII. Curvilinear Square Plots in the Complex Frequency Plane

One useful way to display the behavior of $\tilde{f}(s)$ in the complex frequency plane is by contour plots. By contour plots I mean lines (in general curved) along which some scalar quantity related to $\tilde{f}(s)$ is a constant, usually a real constant. For a real valued function of two variables the contours might be for constant values of the function.

Section II discusses the argument number N_a for a function $\tilde{f}(s)$ based on the complex logarithm as $\ell n(\tilde{f}(s))$ (equation 2.5). The argument number is closely related to zeros and poles of $\tilde{f}(s)$. One might then think that plots of $\ell n(\tilde{f}(s))$ could illustrate some interesting features of $\tilde{f}(s)$.

It is a common practice to plot contours of constant $|\tilde{f}(s)|$ in the s plane as a method of determining the locations of all the zeros of $\tilde{f}(s)$ in a particular part of the s plane.[3] In such cases $\tilde{f}(s)$ is taken as a denominator function $\tilde{d}(s)$, such as a determinant, which typically has no poles in the finite s plane, except possibly at $s = 0$.

The considerations in this note lead one to use $\ell n(\tilde{f}(s))$ in the s plane to better illustrate the pole-zero properties of $\tilde{f}(s)$. Since

$$\ell n(\tilde{f}(s)) = \ell n(|\tilde{f}(s)|) + i \arg(\tilde{f}(s)) \tag{7.1}$$

then let us consider contour plots of $\ell n(|\tilde{f}(s)|)$ and $\arg(\tilde{f}(s))$ in the s plane. Note that contours of constant $\ell n(|\tilde{f}(s)|)$ are also contours of constant $|\tilde{f}(s)|$. Thus contour plots of $\ell n(\tilde{f}(s))$ (real and imaginary parts) effectively include contour plots of $|\tilde{f}(s)|$.

Define quantities

$$\Delta \equiv \frac{2\pi}{\ell} \tag{7.2}$$

$$\ell \equiv \text{a positive integer}$$

Then choose contour values as

$$\ell n(|\tilde{f}(s)|) = 0, \pm\Delta, \pm 2\Delta, \pm 3\Delta, \cdots \tag{7.3}$$

$$\arg(\tilde{f}(s)) = 0, \pm\Delta, \pm 2\Delta, \pm 3\Delta, \cdots$$

where the string of contour values is truncated for convenience on both ends of the string of choices. In the case of the argument contours the truncation is given by the principal value as

$$-\pi < \arg(\tilde{f}(s)) \leq \pi \tag{7.4}$$

Note that the contour choices in equations 7.3 are for equally spaced function values for real and imaginary parts. This results in a curvilinear square plot so that the two sets of contour plots (superimposed) are closely interrelated. Except at zeros, poles, and saddles the two sets of contours intersect at right angles. At zeros and poles $\arg(\tilde{f}(s))$ is undefined; at saddles four or more constant $\arg(\tilde{f}(s_\Sigma))$ contours and four or more constant $|\tilde{f}(s_\Sigma)|$ contours intersect as discussed in section VI.

Note that a first order zero or pole has exactly ℓ constant argument lines leading from it, a second order zero or pole has 2ℓ constant argument lines. Thus this type of plot directly exhibits the order of zeros and poles of $\tilde{f}(s)$.

In addition to the contour values specified in equations 7.3 there are special contour values which can be included in the plot, perhaps as specially marked contours (such as dotted, dashed, etc.). These special contour values are specified by the saddles as discussed in section VI. For contour lines passing through a saddle s_Σ choose

$$\begin{aligned} \ell n(|\tilde{f}(s)|) &= \ell n(|\tilde{f}(s_\Sigma)|) \\ \arg(\tilde{f}(s)) &= \arg(\tilde{f}(s_\Sigma)) \end{aligned} \tag{7.5}$$

Note that not all values of s satisfying equations 7.5 are chosen for contour plots. Only those lying on contours passing through that particular saddle are used.

These special contours, which might be called saddle contours, have special properties. In particular the constant argument saddle contours divide up the s plane in a convenient manner. The constant argument contours (except for the saddle contours) pass from zeros to poles (including the point at infinity). Such constant argument contours do not cross saddle argument contours. Thus the saddle argument contours divide up the complex s plane into regions each of which contains constant argument contours associated with only one zero, or with only one pole. The saddle argument contours come in at least two

types: those which go from saddles to zeros and those which go from saddles to poles (the point at infinity being included for zeros and poles as necessary). The saddle argument contours from saddles to poles divide up the s plane into regions with the constant argument contours associated with a particular zero. The saddle argument contours from saddles to zeros divide up the s plane into regions with the constant argument contours associated with a particular pole.

Another advantage of using $\ell n(\tilde{f}(s))$ for plots is that $\tilde{f}(s)$ often behaves as an exponential function of s. Taking the logarithm of such a function then makes it grow much less radically (in an upper bound sense) as $|s| \to \infty$ in the left half plane. Note that matrices derived from electromagnetic integral equations have elements which get large in the left half plane with forms like e^{-sx} with $x > 0$. Thus the use of $\ell n(\tilde{f}(s))$ has an advantage of being better bounded in the left half of the s plane. Except near zeros and poles $\ell n(\tilde{f}(s))$ is more slowly varying.

The use of $\ell n(\tilde{f}(s))$ for contour plots in the s plane portends well for future SEM investigations. Already the use of constant $\arg(\tilde{f}(s))$ contours gives further insight into the behavior of the impedance loaded loop (TORUS) in a forthcoming note by R. F. Blackburn and of a resistive damping structure for an SGEMP simulator in a forthcoming note by T. L. Brown.

Finally the use of such curvilinear plots of $\ell n(\tilde{f}(s))$ would seem useful for synthesis problems for antennas (EMP simulators) and scatterers. If the denominator function in the response is of the form $\tilde{f}(s) + P$ where P is simply related to a loading impedance then P can be chosen within certain restrictions to shift the zeros of the denominator to more desirable positions in the s plane. Note that $\tilde{f}(s)$ need not be a determinant; it can also be an eigenvalue, as will hopefully be discussed in a future note.

VIII. Changing the Denominator Function Without Changing the Pole-Zero Pattern

Another technique for use in plotting a function in the s plane is to divide by another function which might be thought of as a scaling function. Then define

$$\tilde{f}(s) \equiv \frac{\tilde{d}(s)}{\tilde{d}_s(s)} \tag{8.1}$$

where $\tilde{f}(s)$ is the new function, $\tilde{d}_s(s)$ is our scaling function, and $\tilde{d}(s)$ might be a determinant of interest.

If $\tilde{d}_s(s)$ is required to be an entire function with no zeros it can have the general form[11,15]

$$\tilde{d}_s(s) = e^{\tilde{g}(s)} \tag{8.2}$$

where $\tilde{g}(s)$ is an entire function. As an example suppose that $\tilde{d}(s)$ is the determinant of an N × N matrix where the matrix elements are related to the free space dyadic Green's function evaluated between two zones at $\vec{r}_n$ and $\vec{r}_m$ on the body. The individual matrix elements will then have exponential terms which can be approximated as

$$e^{-\frac{s}{c}|\vec{r}_n - \vec{r}_m|}$$

Since a determinant can be expanded as a sum of products then one might look for the maximum T of the form

$$T = \sum_{j=1}^{N} |\vec{r}_{n(j)} - \vec{r}_{m(j)}| \tag{8.3}$$

found in the determinant. Call this T_{max}. Then one choice is

$$\tilde{g}(s) = -\frac{s}{c} T_{max}$$

$$\tilde{d}_s(s) = e^{-\frac{s}{c} T_{max}} \tag{8.4}$$

By this procedure one should be able to remove the leading exponential term in the determinant, but if there is cancellation of this leading term in the determinant this procedure may not be useful in the above form. Other choices of $\tilde{g}(s)$ are possible. By looking at the behavior of $\tilde{d}(s)$ as $|s| \to \infty$ one can try to pull out a leading exponential term either theoretically or numerically from computed results. Geometrical theory of diffraction (GTD) considerations can help to find leading high frequency terms.

Note that if $\ell n(\tilde{f}(s))$ is used for plots we have

$$\ell n(\tilde{f}(s)) = \ell n(\tilde{d}(s)) - \ell n(\tilde{d}_s(s))$$

$$= \ell n(\tilde{d}(s)) - \tilde{g}(s) \qquad (8.5)$$

Thus consideration of $\ell n(\tilde{d}(s))$ for large $|s|$ can indicate some convenient choices of $\tilde{g}(s)$ and perhaps simplify the problem somewhat.

One purpose of choosing a scaling function is not for plotting, but to improve numerical accuracy in the numerical evaluation of contour integrals, including those for the argument number and the Cauchy integral formula (including for the derivatives with respect to s). Having chosen a contour C one may wish to avoid large variations of $|\tilde{d}(s)|$ around the contour. A scaling function $\tilde{d}_s(s)$ such as discussed above can help in this regard. If $\tilde{d}_s(s)$ is used to help in finding zeros and poles of $\tilde{d}(s)$ (or even saddles of $\tilde{d}(s)$ with some modifications in the procedure) from the numerical approximations to the contour integrals, then $\tilde{d}_s(s)$ need not be included in the plots. Even if $\tilde{d}_s(s)$ is used to help calculate the function values the results for $\tilde{f}(s)$ can be multiplied by $\tilde{d}_s(s)$ to obtain $\tilde{d}(s)$ which is then plotted.

The use of a scaling function $\tilde{d}_s(s)$ can then be helpful in calculating and displaying $\tilde{d}(s)$. However, the scaling function need not appear in the final results. If $\tilde{d}(s)$ has special properties associated with the physical quantity it represents, then one may wish to display $\tilde{d}(s)$ directly without a scaling function.

IX. Summary

Recognizing that the solutions of electromagnetic problems are analytic functions of the complex frequency s except at singularities is the basis of the singularity expansion method (SEM). This analytic property with respect to s allows the use of various complex variable theorems in more efficiently obtaining desired information about the solution. Note that branch cuts and essential singularities are included in the general SEM but they are avoided in the use of certain special techniques involving contour integrals.

In particular the argument number integral and the Cauchy integral formula, including its derivatives with respect to s, can be used to find information about a function $\tilde{f}(s)$ inside the contour from its values on the contour. If $\tilde{f}(s)$ requires lengthy calculations, such as the determinant of a large matrix, then such techniques based on contour integrals can be used to more efficiently find the function values, including its zeros, poles, and saddles. Since $\tilde{f}(s)$ is usually conjugate symmetric (or can often be made so) then additional benefits can be gained for contours near the Re[s] axis by making the contours conjugate symmetric as well to utilize the conjugate symmetry of $\tilde{f}(s)$. In such cases the contour integrals effectively reduce to integrals over a half contour in the upper half s plane.

The use of the argument number integral points out some interesting properties of the singularities of the function $\tilde{f}(s)$ of interest. If $\tilde{f}(s)$ is a continuous function of some parameter P of interest (say an impedance) then as zeros and poles coalesce and split the argument number for a contour surrounding these zeros and poles is conserved. The argument number can then be thought of as a singularity number. This singularity number is conserved much as a quantum number in quantum mechanics. Note that the function is not allowed to be identically zero.

Other points of particular interest are saddles where $\tilde{f}(s) \neq 0$ but $d\tilde{f}/ds = 0$. Using the argument number integral on $d\tilde{f}/ds$ saddles can be found in a manner similar to finding zeros of $\tilde{f}(s)$. Saddles coalesce and split with the saddle number conserved provided zeros and poles are kept out of the process. Including the presence of zeros (of second or higher order) and poles a generalized saddle number can be defined. The generalized saddle number is conserved. Note that $d\tilde{f}/ds$ is not allowed to be identically zero.

Having techniques for calculating and displaying $\tilde{f}(s)$ in terms of $\ell n(\tilde{f}(s))$ and pointing out the zeros, poles, and saddles, one can then try to modify the pole-zero pattern of $\tilde{f}(s)$. If $\tilde{f}(s,P)$ is considered in a simple form such as $\tilde{f}(s) + P$ then by appropriate choice of P the zeros of $\tilde{f}(s,P)$ can be shifted,

hopefully to more desirable positions. This is basically a synthesis procedure for antennas and scatterers. It is especially appropriate for synthesizing the transient or extremely wideband properties of antennas and scatterers. In the form $\tilde{f}(s) + P$ standard root locus procedures apply for this synthesis, much as in the case of circuits or control systems. Note that $\tilde{f}(s)$ need not be a determinant but can be an eigenvalue. As will hopefully be discussed in a future note $\tilde{f}(s)$ can be an eigenvalue and P can be a loading impedance function, making the synthesis decompose according to the eigenfunctions.

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