Mathematics Notes

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Electromagnetic Reciprocity and Energy Theorems for Free Space Including Sources Generalized to Numerous Theorems, to Combined Fields, and to Complex Frequency Domain

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Abstract

This note considers the generalization of the usual Lorentz reciprocity theorem for two sets of electromagnetic fields. This generalization includes combined fields, identical or opposite complex frequencies, energy forms of such equations, and finite or infinite volumes (with retarded and/or advanced solutions). Numerous such relations result and patterns can be seen in the forms of these results.

Foreword

"Now, if you'll only attend, Kitty, and not talk so much, I'll tell you all my ideas about Looking-glass House. First, there's the room you can see through the glass--that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair--all but the bit just behind the fireplace. Oh, I do so wish I could see that bit! I want so much to know whether they've a fire in the winter; you never can tell, you know, unless our fire smokes, and then smoke comes up in that room, too--but that may be only pretense, just to make it look as if they had a fire. Well, then, the books are something like our books, only the words go the wrong way; I know that, because I've held up one of our books to the glass, and then they hold up one in the other room."

Lewis Carroll,
Through the Looking Glass

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I. Introduction

Reciprocity relations in electromagnetic theory have been of interest for quite some time. What is usually referred to as the Lorentz reciprocity theorem is[11,12,13,14]

$$\int_S [\vec{E}^{(1)} \times \vec{H}^{(2)} + \vec{H}^{(1)} \times \vec{E}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V [-\vec{E}^{(1)} \cdot \vec{J}^{(2)} + \vec{H}^{(1)} \cdot \vec{J}_m^{(2)}$$

$$+ \vec{E}^{(2)} \cdot \vec{J}^{(1)} - \vec{H}^{(2)} \cdot \vec{J}_m^{(1)}]dV \qquad (1.1)$$

where V is a volume bounded by a closed surface (or closed surfaces) S with outward pointing unit normal $\vec{n}$. The superscripts (1, 2, etc.) denote particular solutions of Maxwell's equation valid in V. Under certain conditions it can be shown that each side of equation 3.1 is identically zero. This leads to the concept of reaction introduced by Rumsey.[12] The reaction has been defined as

$$\text{reaction} \equiv \int_V [\vec{E}^{(2)} \cdot \vec{J}^{(1)} - \vec{H}^{(2)} \cdot \vec{J}_m^{(1)}]dV \qquad (1.2)$$

where the reaction is between the 2 fields and 1 sources and is the same as the reaction between the 1 sources and the 2 fields.

The purpose of this note is to generalize the above type of relation to various similar equations and to provide a unifying theory for the various relations of this general type obtained. Such relations are in general quite useful for solution of electromagnetic problems. One use is for orthogonality relations between different modes which are combined to represent a solution of Maxwell's equations. One example of this is the orthogonality of modes on a cylindrical transmission line.[1,6] Recently I have been considering the orthogonality of the eigenmodes for general antenna and scattering problems. This orthogonality problem led to the investigation reported here.

The general form of the equations includes not only reciprocity but energy theorems, such as the usual Poynting vector theorem, as well. This note considers combined fields, the case that the 1 and 2 solutions are the same, and the effect of choosing retarded and/or advanced solutions for the fields.

In outline section II considers some preliminary material concerning Maxwell's equations. Sections III through VII consider the reciprocity/energy theorems for cases for which all terms are evaluated at the same complex frequency, s. Section VIII generalizes the basic equation to include various complex frequencies; sections IX through XI apply this to opposite frequencies for the two solutions. Up to this point only finite volumes of space are used for the results. Sections XII and XIII consider some of the terms needed in extending to infinite volumes. Section XIV defines various forms of reaction (for reciprocity theorems) and subreaction (for energy theorems) for use in subsequent sections. Sections XV through XVIII treat the reaction and energy theorems for infinite volumes that result, using various combinations of retarded and advanced fields. Section XIX then summarizes the patterns among the various results obtained.

An important feature of the development is the use of the combined fields as discussed in a previous note.[2] The combined field considerably compacts Maxwell's equations and is used as a starting point for all the cases considered in this note.

The section order reflects the historical development of this note. The order is one of increasing insight into the generality of the reciprocity and energy relations. The fundamental equation is developed in section VII. Reaction and subreaction are defined in section XIV and used for theorems for all space in subsequent sections. Section XIX considers some patterns of the various results considered as a whole.

II. Combined Electromagnetic Quantities

First let us briefly review the combined field and potential development given in another note.[2] The two sided Laplace transform with respect to time t is indicated by a tilde ~ above the quantity; the resulting transform variable is the complex frequency s. The development in this note is oriented toward results in complex frequency domain. These results are a generalization of the usual combined field considered previously.[5,10]

Maxwell's equations are

$$\begin{aligned}
&\nabla \times \tilde{\vec{E}} = -s\mu\tilde{\vec{H}} - \tilde{\vec{J}}_m \\
&\nabla \times \tilde{\vec{H}} = s\varepsilon\tilde{\vec{E}} + \tilde{\vec{J}} \\
&\nabla \cdot \tilde{\vec{B}} = \nabla \cdot (\mu\tilde{\vec{H}}) = \tilde{\rho}_m \\
&\nabla \cdot \tilde{\vec{D}} = \nabla \cdot (\varepsilon\tilde{\vec{E}}) = \tilde{\rho} \\
&\nabla \cdot \tilde{\vec{J}} = -s\tilde{\rho} \\
&\nabla \cdot \tilde{\vec{J}}_m = -s\tilde{\rho}_m
\end{aligned} \tag{2.1}$$

In this general formulation both electric and magnetic current and charge densities are included for generality, as well as usefulness in certain problems. For present purposes we take

$$\mu = \mu_o \,, \qquad \varepsilon = \varepsilon_o \tag{2.2}$$

and lump the effects of dielectric and permeable media into the electric and magnetic current densities. The associated free space propagation constant, propagation speed, and wave impedance are

$$\gamma = \frac{s}{c} \,, \qquad c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \,, \qquad Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{2.3}$$

The electric and magnetic vector and scalar potentials are related to the fields as

$$\tilde{\vec{E}} = -\nabla\tilde{\Phi} - s\tilde{\vec{A}} - \frac{1}{\varepsilon_o} \nabla \times \tilde{\vec{A}}_m$$

$$\tilde{\vec{H}} = \frac{1}{\mu_o} \nabla \times \tilde{\vec{A}} - \nabla\tilde{\Phi}_m - s\tilde{\vec{A}}_m \tag{2.4}$$

and have the Lorentz gauge relations as

$$\nabla \cdot \tilde{\vec{A}} + \frac{s}{c^2} \tilde{\Phi} = 0$$

$$\nabla \cdot \tilde{\vec{A}}_m + \frac{s}{c^2} \tilde{\Phi}_m = 0 \tag{2.5}$$

Wave equations for the fields are

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{E}} = -s\mu_o\tilde{\vec{J}} - \nabla \times \tilde{\vec{J}}_m$$

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{H}} = \nabla \times \tilde{\vec{J}} - s\varepsilon_o\tilde{\vec{J}}_m \tag{2.6}$$

and Helmholtz equations for the potentials are

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}} = -\mu_o\tilde{\vec{J}}$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{\Phi}} = -\frac{1}{\varepsilon_o} \tilde{\rho}$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}}_m = -\varepsilon_o\tilde{\vec{J}}_m \tag{2.7}$$

$$[\nabla^2 - \gamma^2]\tilde{\Phi}_m = -\frac{1}{\mu_o} \tilde{\rho}_m$$

where we have the operator relation

$$\nabla \times \nabla \times \equiv \nabla\nabla \cdot - \nabla^2 \tag{2.8}$$

Define the combined quantities

$$\tilde{\vec{F}}_q \equiv \tilde{\vec{E}} + qiZ_o\tilde{\vec{H}}$$

$$\tilde{\vec{K}}_q \equiv \tilde{\vec{J}} + q\frac{i}{Z_o}\tilde{\vec{J}}_m$$

$$\tilde{Q}_q \equiv \tilde{\rho} + q\frac{i}{Z_o}\tilde{\rho}_m \tag{2.9}$$

$$\tilde{\vec{C}}_q \equiv \tilde{\vec{A}} + qiZ_o\tilde{\vec{A}}_m$$

$$\tilde{\phi}_q \equiv \tilde{\Phi} + qiZ_o\tilde{\Phi}_m$$

where

$$q = \pm 1 \tag{2.10}$$

We have the combined field equations

$$[\nabla \times - qi\gamma]\tilde{\vec{F}}_q = qiZ_o\tilde{\vec{K}}_q$$

$$\nabla \cdot \tilde{\vec{F}}_q = \frac{1}{\varepsilon_o}\tilde{Q}_q \tag{2.11}$$

$$\nabla \cdot \tilde{\vec{K}}_q = -s\tilde{Q}_q$$

The combined field is related to the combined potentials as

$$\tilde{\vec{F}}_q = -\nabla\tilde{\phi}_q + [-s + qic\nabla \times]\tilde{\vec{C}}_q \tag{2.12}$$

with the Lorentz gauge as

$$\nabla \cdot \tilde{\vec{C}}_q + \frac{s}{c^2}\tilde{\phi}_q = 0 \quad . \tag{2.13}$$

The combined field wave equation is

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{F}}_q = -[s\mu_o + qiZ_o\nabla \times]\tilde{\vec{K}}_q \tag{2.14}$$

and the combined potentials Helmholtz equations are

$$\begin{aligned} [\nabla^2 - \gamma^2]\tilde{\vec{C}}_q &= -\mu_o\tilde{\vec{K}}_q \\ [\nabla^2 - \gamma^2]\tilde{\phi}_q &= -\frac{1}{\varepsilon_o}\tilde{Q}_q \end{aligned} \tag{2.15}$$

III. Combined Reciprocity and Energy for Regions of Space with Finite Dimensions

Our starting point is a basic vector analysis relation[7]

$$\nabla \cdot (\vec{a} \times \vec{b}) = (\nabla \times \vec{a}) \cdot \vec{b} - \vec{a} \cdot (\nabla \times \vec{b}) \tag{3.1}$$

This can be applied to two combined fields as

$$\begin{aligned}
\nabla \cdot \left(\tilde{\vec{F}}^{(1)}_{q_1} \times \tilde{\vec{F}}^{(2)}_{q_2}\right) &= \left(\nabla \times \tilde{\vec{F}}^{(1)}_{q_1}\right) \cdot \tilde{\vec{F}}^{(2)}_{q_2} - \tilde{\vec{F}}^{(1)}_{q_1} \cdot \left(\nabla \times \tilde{\vec{F}}^{(2)}_{q_2}\right) \\
&= \left[q_1 i\gamma \tilde{\vec{F}}^{(1)}_{q_1} + q_1 i Z_o \tilde{\vec{K}}^{(1)}_{q_1}\right] \cdot \tilde{\vec{F}}^{(2)}_{q_2} \\
&\quad - \tilde{\vec{F}}^{(1)}_{q_1} \cdot \left[q_2 i\gamma \tilde{\vec{F}}^{(2)}_{q_2} + q_2 i Z_o \tilde{\vec{K}}^{(1)}_{q_2}\right] \\
&= (q_1 - q_2) i\gamma \tilde{\vec{F}}^{(1)}_{q_1} \cdot \tilde{\vec{F}}^{(2)}_{q_2} \\
&\quad - q_2 i Z_o \tilde{\vec{F}}^{(1)}_{q_1} \cdot \tilde{\vec{K}}^{(2)}_{q_2} \\
&\quad + q_1 i Z_o \tilde{\vec{F}}^{(2)}_{q_2} \cdot \tilde{\vec{K}}^{(1)}_{q_1}
\end{aligned} \tag{3.2}$$

where the superscripts refer to particular solutions of the Maxwell equations in some region of space. The subscripts are $q_1 = \pm 1$ applying to the 1 fields etc., and $q_2 = \pm 1$ applying to the 2 fields etc. The pair (q_1, q_2) can take on four sets of values. Equation 3.2 can be thought of as the differential form for combined reciprocity. This set of steps applies to obtaining a differential form for Lorentz reciprocity (equation 1.1) as well. Note that in equation 3.2 the curl terms are replaced by use of the combined Maxwell equations (equations 2.11).

Integrate equation 3.2 over a volume V bounded by a surface S. Then apply the Gauss divergence theorem to the divergence term to obtain a surface integral giving

$$\int_S \left[\tilde{\vec{F}}_{q_1}^{(1)} \times \tilde{\vec{F}}_{q_2}^{(2)} \right] \cdot \vec{n} \, dS$$

$$= \int_V \left[(q_1 - q_2) i\gamma \tilde{\vec{F}}_{q_1}^{(1)} \cdot \tilde{\vec{F}}_{q_2}^{(2)} \right.$$

$$- q_2 i Z_o \tilde{\vec{F}}_{q_1}^{(1)} \cdot \tilde{\vec{K}}_{q_2}^{(2)}$$

$$\left. + q_1 i Z_o \tilde{\vec{F}}_{q_2}^{(2)} \cdot \tilde{\vec{K}}_{q_1}^{(1)} \right] dV \qquad (3.3)$$

where $\vec{n}$ is the unit outward pointing normal on S. This result can be termed the combined reciprocity and energy theorem. It is of quite similar form to the Lorentz reciprocity theorem (equation 1.1) with an additional term. Of course combined fields and combined current densities are used.

Note that as it stands combined reciprocity and energy (equation 3.3) is quite general. As long as the volume V contains no current densities (electric or magnetic) which are not included as "source" current densities this result applies. Furthermore no assumption has yet been made regarding whether the fields are incoming or outgoing waves or some combination of the two; they merely must satisfy Maxwell's equations in V.

The next several sections consider various results deducible from the combined reciprocity and energy theorem. This involves various choices of q_1 and q_2 and for the possible relation of the 1 and 2 combined quantities.

Beginning in section VII this result is generalized to include different complex frequencies. Later beginning in section XII advanced and retarded fields are introduced.

IV. Decomposition of Combined Reciprocity and Energy for Regions of Space with Finite Dimensions

Consider now special combinations of q_1 and q_2 so as to decompose equation 3.3 into various special results.

A. Reciprocity: the case of $q_1 = q_2 \equiv q$

Equation 3.3 becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)} \times \tilde{\vec{F}}_q^{(2)}] \cdot \vec{n}\, dS$$

$$= qiZ_o \int_V [-\tilde{\vec{F}}_q^{(1)} \cdot \tilde{\vec{K}}_q^{(2)} + \tilde{\vec{F}}_q^{(2)} \cdot \tilde{\vec{K}}_q^{(1)}] dV \qquad (4.1)$$

which can be called the combined reciprocity theorem. If we let q = 1, then q = -1, and take the sum and difference of the two results we obtain first

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{E}}^{(2)} - Z_o^2 \tilde{\vec{H}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{J}}_m^{(2)} + Z_o^2 \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{J}}^{(2)}$$

$$- \tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{J}}_m^{(1)} - Z_o^2 \tilde{\vec{H}}^{(2)} \cdot \tilde{\vec{J}}^{(1)}] dV \qquad (4.2)$$

which is just like the Lorentz reciprocity theorem except that the magnetic field instead of the electric field appears with the current density (electric). Let us refer to this as the magnetic reciprocity theorem. Second we have

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{H}}^{(2)} + \tilde{\vec{H}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V [-\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{J}}^{(2)} + \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{J}}_m^{(2)}$$

$$+ \tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{J}}^{(1)} - \tilde{\vec{H}}^{(2)} \cdot \tilde{\vec{J}}_m^{(1)}] dV \qquad (4.3)$$

which is the Lorentz reciprocity theorem previously quoted. Let us refer to this as the electric reciprocity theorem.

B. Energy: the case of $q_1 = -q_2 \equiv q$

Equation 3.3 becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)} \times \tilde{\vec{F}}_{-q}^{(2)}] \cdot \vec{n}\, dS$$

$$= qi\int_V [2\gamma\tilde{\vec{F}}_q^{(1)} \cdot \tilde{\vec{F}}_{-q}^{(2)} + Z_o\tilde{\vec{F}}_q^{(1)} \cdot \tilde{\vec{K}}_{-q}^{(2)}$$

$$+ Z_o\tilde{\vec{F}}_{-q}^{(2)} \cdot \tilde{\vec{K}}_q^{(1)}]dV \qquad (4.4)$$

which can be called the combined energy theorem. Letting q = 1, then q = -1, and taking the sum and difference gives first

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{E}}^{(2)} + Z_o^2\tilde{\vec{H}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \Big\{2\gamma Z_o [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{H}}^{(2)} - \tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{H}}^{(1)}]$$

$$+ [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{J}}_m^{(2)} - Z_o^2\tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{J}}^{(2)}]$$

$$+ [-\tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{J}}_m^{(1)} + Z_o^2\tilde{\vec{H}}^{(2)} \cdot \tilde{\vec{J}}^{(1)}]\Big\} dV \qquad (4.5)$$

Note that the dot products of the fields have electric and magnetic fields appearing together; this is an energy-like combination. Since it involves both two different Maxwell equation solutions (mixed solutions) and electric and magnetic fields combined in the dot products (as well as $\tilde{\vec{H}} \cdot \tilde{\vec{J}}$, $\tilde{\vec{E}} \cdot \tilde{\vec{J}}_m$, $\tilde{\vec{E}} \times \tilde{\vec{E}}$, and $\tilde{\vec{H}} \times \tilde{\vec{H}}$ terms) which might be referred to as interchanged quantities, then this result might be called the interchanged mixed energy theorem. Second we have

$$\int_S [-\tilde{\vec{E}}^{(1)} \times \tilde{\vec{H}}^{(2)} + \tilde{\vec{H}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \Big\{ 2 \frac{\gamma}{Z_o} [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{E}}^{(2)} + Z_o^2 \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{H}}^{(2)}]$$

$$+ [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{J}}^{(2)} + \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{J}}_m^{(2)}]$$

$$+ [\tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{J}}^{(1)} + \tilde{\vec{H}}^{(2)} \cdot \tilde{\vec{J}}_m^{(1)}] \Big\} dV \qquad (4.6)$$

which can be referred to as the mixed energy theorem. This is basically the Poynting vector theorem for mixed solutions in complex frequency domain. Note that complex conjugates are not used here.

C. Combinations of previous results

Equations 4.2 and 4.5 can be combined to give the two results

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \Big\{ \gamma Z_o [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{H}}^{(2)} - \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{E}}^{(2)}]$$

$$+ [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{J}}_m^{(2)} - \tilde{\vec{E}}^{(2)} \cdot \tilde{\vec{J}}_m^{(1)}] \Big\} dV \qquad (4.7)$$

and

$$\int_S [\tilde{\vec{H}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \Big\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{H}}^{(2)} - \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{E}}^{(2)}]$$

$$+ [-\tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{J}}^{(2)} + \tilde{\vec{H}}^{(2)} \cdot \tilde{\vec{J}}^{(1)}] \Big\} dV \qquad (4.8)$$

Note that some of the terms cancelled out in the process.

Equations 4.3 and 4.6 can be combined to give

$$\int_S [\vec{\tilde{E}}^{(1)} \times \vec{\tilde{H}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \left\{ -\frac{\gamma}{Z_o} [\vec{\tilde{E}}^{(1)} \cdot \vec{\tilde{E}}^{(2)} + Z_o^2 \vec{\tilde{H}}^{(1)} \cdot \vec{\tilde{H}}^{(2)}] \right.$$

$$\left. + [-\vec{\tilde{E}}^{(1)} \cdot \vec{\tilde{J}}^{(2)} - \vec{\tilde{H}}^{(2)} \cdot \vec{\tilde{J}}_m^{(1)}] \right\} dV \qquad (4.9)$$

and

$$\int_S [\vec{\tilde{H}}^{(1)} \times \vec{\tilde{E}}^{(2)}] \cdot \vec{n}\, dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\vec{\tilde{E}}^{(1)} \cdot \vec{\tilde{E}}^{(2)} + Z_o^2 \vec{\tilde{H}}^{(1)} \cdot \vec{\tilde{H}}^{(2)}] \right.$$

$$\left. + [\vec{\tilde{H}}^{(1)} \cdot \vec{\tilde{J}}_m^{(2)} + \vec{\tilde{E}}^{(2)} \cdot \vec{\tilde{J}}^{(1)}] \right\} dV \qquad (4.10)$$

These two equations can be regarded as the same since they merely interchange the 1 and 2 superscripts.

D. The case of $s = 0$

Now let $s = 0$. The results of this section can be readily reduced to this special case assuming that the field and current density integrals are $o(1/s)$ as $s \to 0$ after γ has been factored out wherever it appears. In other words it is required for the simple results that as $\gamma \to 0$ the integrals of γ times the product of two fields be $o(1)$, i.e. tend to 0 as $\gamma \to 0$.

The results of section IV A, the reciprocity theorems, are not affected by setting $\gamma = 0$ in this sense. However, the results of sections IV B and IV C are somewhat simplified in that the terms involving dot products of two fields can be all set to zero; these are basically the field "energy density" terms times γ.

V. Combined Reciprocity and Energy for No Currents Within the Volume of Interest

Now let $\vec{\tilde{J}}$ and $\vec{\tilde{J}}_m$ be identically zero in V. The combined reciprocity and energy theorem (equation 3.3) becomes

$$\int_S [\vec{\tilde{F}}_{q_1} \times \vec{\tilde{F}}_{q_2}] \cdot \vec{n} dS = (q_1 - q_2) i\gamma \int_V \vec{\tilde{F}}_{q_1} \cdot \vec{\tilde{F}}_{q_2} dV \tag{5.1}$$

A. Reciprocity: the case of $q_1 = q_2 \equiv q$

The combined reciprocity theorem (equation 4.1) becomes

$$\int_S [\vec{\tilde{F}}_q \times \vec{\tilde{F}}_q] \cdot \vec{n} dS = 0 \tag{5.2}$$

The magnetic reciprocity theorem (equation 4.2) becomes

$$\int_S [\vec{\tilde{E}}^{(1)} \times \vec{\tilde{E}}^{(2)} - Z_o^2 \vec{\tilde{H}}^{(1)} \times \vec{\tilde{H}}^{(2)}] \cdot \vec{n} dS = 0 \tag{5.3}$$

and the electric reciprocity theorem (equation 4.3) becomes

$$\int_S [\vec{\tilde{E}}^{(1)} \times \vec{\tilde{H}}^{(2)} + \vec{\tilde{H}}^{(1)} \times \vec{\tilde{E}}^{(2)}] \cdot \vec{n} dS = 0 \tag{5.4}$$

Note that these reciprocity theorems involve only surface integrals and are thus appropriate for considering orthogonality of various fields (e.g. modes) on S.

B. Energy: the case of $q_1 = -q_2 \equiv q$

The combined energy theorem (equation 4.4) becomes

$$\int_S [\vec{\tilde{F}}_q^{(1)} \times \vec{\tilde{F}}_{-q}^{(2)}] \cdot \vec{n} dS = qi2\gamma \int_V \vec{\tilde{F}}_q^{(1)} \cdot \vec{\tilde{F}}_{-q}^{(2)} dV \tag{5.5}$$

The interchanged mixed energy theorem (equation 4.5) becomes

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{E}}^{(2)} + Z_o^2 \tilde{\vec{H}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n} dS$$

$$= 2\gamma Z_o \int_V [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{H}}^{(2)} - \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{E}}^{(2)}] dV \qquad (5.6)$$

The mixed energy theorem (equation 4.6) becomes

$$\int_S [-\tilde{\vec{E}}^{(1)} \times \tilde{\vec{H}}^{(2)} + \tilde{\vec{H}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n} dS$$

$$= 2 \frac{\gamma}{Z_o} \int_V [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{E}}^{(2)} + Z_o^2 \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{H}}^{(2)}] dV \qquad (5.7)$$

C. Combinations of previous results

Equations 4.7 and 4.8 give the same results for no current densities in V as

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n} dS = Z_o^2 \int_S [\tilde{\vec{H}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n} dS$$

$$= \gamma Z_o \int_V [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{H}}^{(2)} - \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{E}}^{(2)}] dV \qquad (5.8)$$

Similarly equations 4.9 and 4.10 give the same results for no current densities in V as

$$\int_S [\tilde{\vec{E}}^{(1)} \times \tilde{\vec{H}}^{(2)}] \cdot \vec{n} dS = \int_S [-\tilde{\vec{H}}^{(1)} \times \tilde{\vec{E}}^{(2)}] \cdot \vec{n} dS$$

$$= -\frac{\gamma}{Z_o} \int_V [\tilde{\vec{E}}^{(1)} \cdot \tilde{\vec{E}}^{(2)} + Z_o^2 \tilde{\vec{H}}^{(1)} \cdot \tilde{\vec{H}}^{(2)}] dV \qquad (5.9)$$

VI. Reduction to the Case That the Two Sets of Fields and Current Densities are the Same

Now let the 1 and 2 solutions of Maxwell's equations be identically the same and drop the superscripts to indicate this. Let q_1 and q_2 still be independent, however.

The basic equation, equation 3.3, then takes the form

$$\int_S [\tilde{\vec{F}}_{q_1} \times \tilde{\vec{F}}_{q_2}] \cdot \vec{n}dS$$

$$= \int_V [(q_1 - q_2) i\gamma \tilde{\vec{F}}_{q_1} \cdot \tilde{\vec{F}}_{q_2}$$

$$- q_2 i Z_o \tilde{\vec{F}}_{q_1} \cdot \tilde{\vec{K}}_{q_2}$$

$$+ q_1 i Z_o \tilde{\vec{F}}_{q_2} \cdot \tilde{\vec{K}}_{q_1}]dV \qquad (6.1)$$

Note that since the cross product of a field with itself is zero we have

$$\int_S [\tilde{\vec{F}}_{q_1} \times \tilde{\vec{F}}_{q_2}] \cdot \vec{n}dS$$

$$= (q_2 - q_1) i Z_o \int_S [\tilde{\vec{E}} \times \tilde{\vec{H}}] \cdot \vec{n}dS \qquad (6.2)$$

which simplifies matters somewhat.

A. Reciprocity: the case of $q_1 = q_2 \equiv q$

This case reduces to a trivial result with both sides of each of equations 4.1, 4.2, and 4.3 being identically zero. Evidently reciprocity of a field with respect to itself provides little information.

B. Energy: the case of $q_1 = -q_2 \equiv q$

The combined energy theorem, equation 4.4, becomes

$$\int_S [\tilde{\vec{F}}_q \times \tilde{\vec{F}}_{-q}] \cdot \vec{n} dS$$

$$= qi \int_V [2\gamma \tilde{\vec{F}}_q \cdot \tilde{\vec{F}}_{-q} + Z_o \tilde{\vec{F}}_q \cdot \tilde{\vec{K}}_{-q}$$

$$+ Z_o \tilde{\vec{F}}_{-q} \cdot \tilde{\vec{K}}_q] dV \tag{6.3}$$

Considering the interchanged mixed energy theorem, equation 4.5, note that for both sets of fields the same the terms in the equation are zero or cancel with other terms. The mixed energy theorem, equation 4.6, when applied to a single solution of Maxwell's equations becomes

$$-\int_S [\tilde{\vec{E}} \times \tilde{\vec{H}}] \cdot \vec{n} dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}} \cdot \tilde{\vec{E}} + Z_o^2 \tilde{\vec{H}} \cdot \tilde{\vec{H}}] \right.$$

$$\left. + [\tilde{\vec{E}} \cdot \tilde{\vec{J}} + \tilde{\vec{H}} \cdot \tilde{\vec{J}}_m] \right\} dV \tag{6.4}$$

This can be referred to as the energy theorem for a single solution of Maxwell's equations. This is a Poynting vector theorem in complex frequency domain. Note that no restriction has yet been made regarding advanced and retarded solutions since this can apply to a rather general finite size volume V. Typical forms of the Poynting vector theorem have some of the terms in the form of complex conjugates.[8,9] The present result (equation 6.4) is a valid but different result. Other forms are treated later using different complex frequencies as arguments of the various quantities and/or using advanced and retarded fields.

C. Combinations of previous results

In this section with both sets of fields the same the only non trivial result is that expressed in equation 6.4. Equations 6.2 and 6.1 are equivalent representations of this result.

D. The case of s = 0

For zero complex frequency which implies $\gamma = 0$ equation 6.4 simplifies somewhat in that the only remaining terms are those involving dot products of fields with current densities.

VII. Generalization of Results to Include Different Complex Frequencies

Sections II through VI treat the case that the fields, current densities, etc. are all evaluated at the same complex frequency s. However, this is not a necessary restriction. Consider the following form of Maxwell's equations as

$$
\begin{aligned}
\nabla \times \tilde{\vec{E}}(s) &= -s\mu(s)\tilde{\vec{H}}(s) - \tilde{\vec{J}}_m(s') \\
\nabla \times \tilde{\vec{H}}(s) &= s\varepsilon(s)\tilde{\vec{E}}(s) + \tilde{\vec{J}}(s') \\
\nabla \cdot \tilde{\vec{B}}(s) &= \nabla \cdot (\mu(s)\tilde{\vec{H}}(s)) = \frac{s'}{s}\tilde{\rho}_m(s') \\
\nabla \cdot \tilde{\vec{D}}(s) &= \nabla \cdot (\varepsilon(s)\tilde{\vec{E}}(s)) = \frac{s'}{s}\tilde{\rho}(s') \\
\nabla \cdot \tilde{\vec{J}}(s') &= -s'\tilde{\rho}(s') \\
\nabla \cdot \tilde{\vec{J}}_m(s') &= -s'\tilde{\rho}_m(s')
\end{aligned}
\tag{7.1}
$$

where the "sources," the current and charge densities, are treated as functions of a complex variable s', while the remainder of the terms are treated as functions of s. Actually in this form the fields are functions of both s and s', but the s' is not listed. For present purposes we have

$$
\gamma = \frac{s}{c}, \qquad \gamma' = \frac{s'}{c}, \qquad \varepsilon(s) = \varepsilon_o, \qquad \mu(s) = \mu_o \tag{7.2}
$$

This splitting into s and s' will have use when considering separate frequencies for fields and current densities and letting the surface S tend to infinity.

The scalar and vector potentials have the same relations to the fields and to each other as in equations 2.4, 2.5, 2.12 and 2.13. The wave and Helmholtz equations become

$$
[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{E}}(s) = -s\mu_o\tilde{\vec{J}}(s') - \nabla \times \tilde{\vec{J}}_m(s')
$$

$$
[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{H}}(s) = \nabla \times \tilde{\vec{J}}(s) - s\varepsilon_o\tilde{\vec{J}}_m(s')
$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}}(s) = -\mu_o \tilde{\vec{J}}(s') \tag{7.3}$$

$$[\nabla^2 - \gamma^2]\tilde{\Phi}(s) = -\frac{1}{\varepsilon_o}\frac{s'}{s}\tilde{\rho}(s')$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{A}}_m(s) = -\varepsilon_o \tilde{\vec{J}}_m(s')$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{\Phi}}_m(s) = -\frac{1}{\mu_o}\frac{s'}{s}\tilde{\rho}_m(s')$$

The combined forms of the Maxwell equations are

$$[\nabla \times - qi\gamma]\tilde{\vec{F}}_q(s) = qiZ_o \tilde{\vec{K}}_q(s')$$

$$\nabla \cdot \tilde{\vec{F}}_q(s) = \frac{1}{\varepsilon_o}\frac{s'}{s}\tilde{Q}_q(s') \tag{7.4}$$

$$\nabla \cdot \tilde{\vec{K}}(s') = -s'\tilde{Q}_q(s')$$

and the combined forms of the wave and Helmholtz equations are

$$[\nabla \times \nabla \times + \gamma^2]\tilde{\vec{F}}_q(s) = -[s\mu_o + qiZ_o \nabla \times]\tilde{\vec{K}}_q(s')$$

$$[\nabla^2 - \gamma^2]\tilde{\vec{C}}_q(s) = -\mu_o \tilde{\vec{K}}_q(s') \tag{7.5}$$

$$[\nabla^2 - \gamma^2]\tilde{\phi}_q(s) = -\frac{1}{\varepsilon_o}\frac{s'}{s}\tilde{Q}_q(s')$$

Now consider two combined fields and current densities with superscripts 1 and 2 as functions of complex frequencies s and s' with subscripts 1 and 2 respectively. Apply the vector identity in equation 3.1 and follow the steps in equations 3.2 and 3.3 to give

$$\int_S [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n} dS$$

$$= \int_V [(q_1 i\gamma_1 - q_2 i\gamma_2)\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{F}}^{(2)}_{q_2}(s_2)$$

$$+ q_1 i Z_o \tilde{\vec{F}}^{(2)}_{q_2}(s_2) \cdot \tilde{\vec{K}}^{(1)}_{q_1}(s_1')$$

$$- q_2 i Z_o \tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{K}}^{(2)}_{q_2}(s_2')] dV \tag{7.6}$$

In this generalized form one can choose s_1' and s_2' and then s_1 and s_2 and calculate the resulting fields from given current densities. Many choices of the relations of the various complex frequencies are possible; this note considers some of them. Sections II through V have already considered the case of $s_1 = s_1' = s_2 = s_2'$.

In its present form equation 7.6 might be termed the mixed frequencies combined reciprocity and energy theorem.

VIII. Combined Reciprocity and Energy for Opposite Frequencies for Regions of Space with Finite Dimensions

Considering the general result in equation 7.6 for a mixed frequency combined reciprocity and energy theorem one would like to consider useful special cases. One way to define such special cases is to notice the combinations of terms and try to choose the available parameters to maximize cancellation of terms or more generally to maximize symmetry in the form of the resulting equation.

Sections III through VI consider the case of $s_1 = s_1' = s_2 = s_2'$. As illustrated in equation 3.3 such a choice does simplify the form somewhat by allowing γ to be common to two terms on the right hand side. In this section let us consider another case with similar simplicity. Specifically choose

$$s_1 = s_1' = -s_2 = -s_2' \equiv s \tag{8.1}$$

This gives a result of the form

$$\begin{aligned}\int_S [\vec{\tilde{F}}^{(1)}_{q_1}(s) \times \vec{\tilde{F}}^{(2)}_{q_2}(-s)] \cdot \vec{n}dS \\ = \int_V [(q_1 + q_2) i\gamma \vec{\tilde{F}}^{(1)}_{q_1}(s) \cdot \vec{\tilde{F}}^{(2)}_{q_2}(-s) \\ - q_2 i Z_o \vec{\tilde{F}}^{(1)}_{q_1}(s) \cdot \vec{\tilde{K}}^{(2)}_{q_2}(-s) \\ + q_1 i Z_o \vec{\tilde{F}}^{(2)}_{q_2}(-s) \cdot \vec{\tilde{K}}^{(1)}_{q_1}(s)] dV\end{aligned} \tag{8.2}$$

Note the switch from $q_1 - q_2$ to $q_1 + q_2$ in the leading term on the right hand side in comparing equations 3.3 and 8.2. This will switch the role of the relative values of q_1 and q_2 in what we consider as reciprocity and energy theorems. This result can be termed the opposite frequencies combined reciprocity and energy theorem.

One of the important characteristics of this choice of the two complex frequencies is that they are opposite, i.e., the negative of each other. This has a similarity to a complex conjugate relation. Denoting complex conjugate by $\bar{}$ over the quantity we have for

$$s = \Omega + i\omega \tag{8.3}$$

the relation for $\Omega = 0$

$$s_2 = -s_1 \tag{8.4}$$

is the same as

$$s_2 = \bar{s}_1 \tag{8.5}$$

since

$$\begin{aligned} s_2 &= i\omega_2 \\ s_1 &= i\omega_1 \\ \bar{s}_2 &= -i\omega_2 \\ \omega_1 &= -\omega_2 \end{aligned} \tag{8.6}$$

If we restrict our attention to Laplace transformed quantities, say $\tilde{\vec{X}}$, corresponding to real valued time functions then we have

$$\overline{\tilde{\vec{X}}}(s) = \tilde{\vec{X}}(\bar{s}) \tag{8.7}$$

For $\mathrm{Re}[s] = 0$ this reduces to

$$\tilde{\vec{X}}(-s) = \tilde{\vec{X}}(\bar{s}) = \overline{\tilde{\vec{X}}}(s) \tag{8.8}$$

In working with time harmonic fields ($s = i\omega$) conjugate fields have often been used. Unfortunately conjugation is not an analytic operation. Transforming s to -s is an analytic operation. As the above development indicates this transformation of s to -s is a generalization of conjugation which is analytic in the s plane for real valued time functions.

For complex valued time functions such as $\vec{F}_q(t)$ one must be more careful. Since we have

$$\vec{F}_q(t) = \vec{E}(t) + qiZ_o\vec{H}(t)$$

$$\tilde{\vec{F}}_q(s) = \tilde{\vec{E}}(s) + qiZ_o\tilde{\vec{H}}(s) \tag{8.9}$$

then for $s = i\omega$ we have

$$\begin{aligned} \tilde{\vec{F}}_q(-s) &= \tilde{\vec{E}}(-s) + qiZ_o\tilde{\vec{H}}(-s) \\ &= \overline{\tilde{\vec{E}}}(s) + qiZ_o\overline{\tilde{\vec{H}}}(s) \\ &= \overline{\tilde{\vec{E}}}(s) - \overline{qiZ_o\tilde{\vec{H}}(s)} \\ &= \overline{\tilde{\vec{F}}}_{-q}(s) \end{aligned} \tag{8.10}$$

so that the sign of q is switched in the process.

Equation 8.2 can be readily generalized if one desires to leave the complex frequencies s_1' and s_2' somewhat general. Alternately one might replace the choices in equation 8.1 by

$$s_1 = -s_2 = s \ , \qquad s_1' = -s_2' = s' \tag{8.11}$$

The choice in equation 8.1 is quite interesting, however, as it stands.

IX. Decomposition of Combined Reciprocity and Energy for Opposite Frequencies for Regions of Space with Finite Dimensions

A. Energy for opposite frequencies: the case of $q_1 = q_2 \equiv q$

Equation 8.2 becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)}(s) \times \tilde{\vec{F}}_q^{(2)}(-s)] \cdot \vec{n}dS$$

$$= qi\int_V [2\gamma\tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{F}}_q^{(2)}(-s) - Z_o\tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{K}}_q^{(2)}(-s)$$

$$+ Z_o\tilde{\vec{F}}_q^{(2)}(-s) \cdot \tilde{\vec{K}}_q^{(1)}(s)]dV \qquad (9.1)$$

which can be called the opposite frequencies combined energy theorem. Note the similarity to equation 4.4. Letting $q = +1$ and then $q = -1$, and then adding and subtracting the results gives first

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s) - Z_o^2\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{2\gamma Z_o[-\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s) - \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s)]$$

$$+ [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{J}}_m^{(2)}(-s) + Z_o^2\tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{J}}^{(2)}(-s)]$$

$$+ [-\tilde{\vec{E}}^{(2)}(-s) \cdot \tilde{\vec{J}}_m^{(1)}(s) - Z_o^2\tilde{\vec{H}}^{(2)}(-s) \cdot \tilde{\vec{J}}^{(1)}(s)]\Big\}dV \qquad (9.2)$$

which can be referred to as the opposite frequency interchanged mixed energy theorem. This result is similar to equation 4.5. Next we have

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s) + \tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n} dS$$

$$= \int_V \Big\{ 2 \frac{\gamma}{Z_o} [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s)]$$

$$+ [-\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{J}}^{(2)}(-s) + \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{J}}_m^{(2)}(-s)]$$

$$+ [\tilde{\vec{E}}^{(2)}(-s) \cdot \tilde{\vec{J}}^{(1)}(s) - \tilde{\vec{H}}^{(2)}(-s) \cdot \tilde{\vec{J}}_m^{(1)}(s)] \Big\} dV \qquad (9.3)$$

which can be called the opposite frequency mixed energy theorem. This is similar to equation 4.6.

B. Reciprocity for opposite frequencies: the case of $q_1 = -q_2 \equiv q$

Equation 8.2 becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)}(s) \times \tilde{\vec{F}}_{-q}^{(2)}(-s)] \cdot \vec{n} dS$$

$$= qiZ_o \int_V [\tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{K}}_{-q}^{(2)}(-s) + \tilde{\vec{F}}_{-q}^{(2)}(-s) \cdot \tilde{\vec{K}}_q^{(1)}(s)] dV \qquad (9.4)$$

Note the similarity to equation 4.1. This might be called the opposite frequencies combined reciprocity theorem. Letting $q = +1$ and then $q = -1$, and then adding and subtracting the results gives first

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s) + Z_o^2 \tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n} dS$$

$$= \int_V [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{J}}_m^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{J}}^{(2)}(-s)$$

$$- \tilde{\vec{E}}^{(2)}(-s) \cdot \tilde{\vec{J}}_m^{(1)}(s) + Z_o^2 \tilde{\vec{H}}^{(2)}(-s) \cdot \tilde{\vec{J}}^{(1)}(s)] dV \qquad (9.5)$$

Note the similarity to equation 4.2. This might be termed the opposite frequencies magnetic reciprocity theorem. Second we have

$$\int_S [-\vec{\tilde{E}}^{(1)}(s) \times \vec{\tilde{H}}^{(2)}(-s) + \vec{\tilde{H}}^{(1)}(s) \times \vec{\tilde{E}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V [\vec{\tilde{E}}^{(1)}(s) \cdot \vec{\tilde{J}}^{(2)}(-s) + \vec{\tilde{H}}^{(1)}(s) \cdot \vec{\tilde{J}}_m^{(2)}(-s)$$

$$+ \vec{\tilde{E}}^{(2)}(-s) \cdot \vec{\tilde{J}}^{(1)}(s) + \vec{\tilde{H}}^{(2)}(-s) \cdot \vec{\tilde{J}}_m^{(1)}(s)]dV \quad (9.6)$$

which can be called the opposite frequencies electric reciprocity theorem. Note the similarity to equation 4.3.

C. Combinations of previous results

Equations 9.2 and 9.5 can be combined to give the two results

$$\int_S [\vec{\tilde{E}}^{(1)}(s) \times \vec{\tilde{E}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{ \gamma Z_o [-\vec{\tilde{E}}^{(1)}(s) \cdot \vec{\tilde{H}}^{(2)}(-s) - \vec{\tilde{H}}^{(1)}(s) \cdot \vec{\tilde{E}}^{(2)}(-s)]$$

$$+ [\vec{\tilde{E}}^{(1)}(s) \cdot \vec{\tilde{J}}_m^{(2)}(-s) - \vec{\tilde{E}}^{(2)}(-s) \cdot \vec{\tilde{J}}_m^{(1)}(s)] \Big\} dV \quad (9.7)$$

which is similar to equation 4.7 and

$$\int_S [\vec{\tilde{H}}^{(1)}(s) \times \vec{\tilde{H}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{ \frac{\gamma}{Z_o} [\vec{\tilde{E}}^{(1)}(s) \cdot \vec{\tilde{H}}^{(2)}(-s) + \vec{\tilde{H}}^{(1)}(s) \cdot \vec{\tilde{E}}^{(2)}(-s)]$$

$$+ [-\vec{\tilde{H}}^{(1)}(s) \cdot \vec{\tilde{J}}^{(2)}(-s) + \vec{\tilde{H}}^{(2)}(-s) \cdot \vec{\tilde{J}}^{(1)}(s)] \Big\} dV \quad (9.8)$$

which is similar to equation 4.8.

Equations 9.3 and 9.6 can be combined to give

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s)] \right.$$

$$\left. + [-\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{J}}^{(2)}(-s) - \tilde{\vec{H}}^{(2)}(-s) \cdot \tilde{\vec{J}}_m^{(1)}(s)] \right\} dV \qquad (9.9)$$

which can be compared to equation 4.9 and

$$\int_S [\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s)] \right.$$

$$\left. + [\tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{J}}_m^{(2)}(-s) + \tilde{\vec{E}}^{(2)}(-s) \cdot \tilde{\vec{J}}^{(1)}(s)] \right\} dV \qquad (9.10)$$

which can be compared to equation 4.10. Note that if the indexes "1" and "2" are interchanged and s and -s are interchanged then equation 9.10 is the same as equation 9.9.

D. The case of s = 0

Let s = 0 and assume that the field and current density integrals are $o(1/s)$ as $s \to 0$ after γ has been factored out wherever it appears. The results then simplify considerably because of the removal of various terms.

An additional simplification of setting s = 0 is that for this special case s = -s. This allows direct comparison of the results for s = 0 between section IV and this section. Note the following correlation of equation numbers that results:

Sections III and IV	Sections VIII and IX
3.3	8.2
4.1	9.1
4.2	9.2
4.3	9.3
4.4	9.4
4.5	9.5
4.6	9.6
4.7	9.7
4.8	9.8
4.9	9.9
4.10	9.10

Table 9.1 Equations which are the same if $s = 0$

This serves as a check on the results.

X. Combined Reciprocity and Energy for Opposite Frequencies for No Currents within the Volume of Interest

Letting $\tilde{\vec{J}}$ and $\tilde{\vec{J}}_m$ be identically zero in V the results of sections VIII and IX simplify somewhat. The opposite frequencies combined reciprocity and energy theorem (equation 8.2) becomes

$$\int_S [\tilde{\vec{F}}_{q_1}^{(1)}(s) \times \tilde{\vec{F}}_{q_2}^{(2)}(-s)] \cdot \vec{n} dS = (q_1 + q_2) i\gamma \int_V \tilde{\vec{F}}_{q_1}^{(1)}(s) \cdot \tilde{\vec{F}}_{q_2}^{(2)}(-s) dV \tag{10.1}$$

A. Energy for opposite frequencies: the case of $q_1 = q_2 \equiv q$

The opposite frequencies combined energy theorem (equation 9.1) becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)}(s) \times \tilde{\vec{F}}_q^{(2)}(-s)] \cdot \vec{n} dS = qi2\gamma \int_V \tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{F}}_q^{(2)}(-s) dV \tag{10.2}$$

The opposite frequency interchanged mixed energy theorem (equation 9.2) becomes

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n} dS$$

$$= -2\gamma Z_o \int_V [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s) + \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s)] dV \tag{10.3}$$

The opposite frequency mixed energy theorem (equation 9.3) becomes

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s) + \tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n} dS$$

$$= 2 \frac{\gamma}{Z_o} \int_V [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s)] dV \tag{10.4}$$

B. Reciprocity for opposite frequencies: the case of $q_1 = -q_2 \equiv q$

The opposite frequencies combined reciprocity theorem (equation 9.4) becomes

$$\int_S [\tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{F}}_{-q}^{(2)}(-s)] \cdot \vec{n}dS = \vec{0} \tag{10.5}$$

The opposite frequencies magnetic reciprocity theorem (equation 9.5) becomes

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s) + Z_o^2\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n}dS = \vec{0} \tag{10.6}$$

The opposite frequencies electric reciprocity theorem (equation 9.6) becomes

$$\int_S [-\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s) + \tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n}dS = \vec{0} \tag{10.7}$$

Note that these opposite frequencies reciprocity theorems involve only surface integrals and are appropriate for considering orthogonality of various modes evaluated at opposite frequencies on surfaces of interest.

C. Combinations of previous results

Equations 9.7 and 9.8 give the same results for no current densities in V as

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n}dS = -Z_o^2 \int_S [\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \gamma Z_o \int_V [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{H}}^{(2)}(-s) - \tilde{\vec{H}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s)]dV \tag{10.8}$$

Equations 9.9 and 9.10 give the same results for no current density in V as

$$\int_S [\tilde{\vec{E}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)] \cdot \vec{n}dS = \int_S [\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{E}}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= \frac{\gamma}{Z_o} \int_V [\tilde{\vec{E}}^{(1)}(s) \cdot \tilde{\vec{E}}^{(2)}(-s) - Z_o^2\tilde{\vec{H}}^{(1)}(s) \times \tilde{\vec{H}}^{(2)}(-s)]dV \tag{10.9}$$

XI. Case that the Two Sets of Fields and Current Densities are the Same Except for Opposite Frequencies

Let the 1 and 2 solutions of Maxwell's equations be the same except for being evaluated at opposite frequencies. Accordingly drop the superscripts. Let q_1 and q_2 still be independent.

The basic equation for opposite frequencies, equation 8.2 then becomes

$$\int_S [\tilde{\vec{F}}_{q_1}(s) \times \tilde{\vec{F}}_{q_2}(-s)] \cdot \vec{n}dS$$

$$= \int_V [(q_1 + q_2) i\gamma \tilde{\vec{F}}_{q_1}(s) \cdot \tilde{\vec{F}}_{q_2}(-s)$$

$$- q_2 i Z_o \tilde{\vec{F}}_{q_1}(s) \cdot \tilde{\vec{K}}_{q_2}(-s)$$

$$+ q_1 i Z_o \tilde{\vec{F}}_{q_2}(-s) \cdot \tilde{\vec{K}}_{q_1}(s)]dV \qquad (11.1)$$

The cross product term can be expanded as

$$\int_S [\tilde{\vec{F}}_{q_1}(s) \times \tilde{\vec{F}}_{q_2}(-s)] \cdot \vec{n}dS$$

$$= \int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{E}}(-s) - q_1 q_2 Z_o^2 \tilde{\vec{H}}(s) \times \tilde{\vec{H}}(-s)$$

$$+ q_2 i Z_o \tilde{\vec{E}}(s) \times \tilde{\vec{H}}(-s) + q_1 i Z_o \tilde{\vec{H}}(-s) \times \tilde{\vec{E}}(s)] \cdot \vec{n}dS \qquad (11.2)$$

Unlike equation 6.2 none of the terms cancel. This result will lead to more theorems than in section VI.

A. Energy for opposite frequencies: the case of $q_1 = q_2 \equiv q$

The opposite frequencies combined energy theorem, equation 9.1, becomes

$$\int_S [\tilde{\vec{F}}_q(s) \times \tilde{\vec{F}}_q(-s)] \cdot \vec{n}dS$$

$$= qi\int_V [2\gamma\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{F}}_q(-s) - Z_o\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{K}}_q(-s)$$

$$+ Z_o\tilde{\vec{F}}_q(-s) \cdot \tilde{\vec{K}}_q(s)]dV \qquad (11.3)$$

The opposite frequencies interchanged energy theorem, from equation 9.2, is

$$\int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{E}}(-s) - Z_o^2\tilde{\vec{H}}(s) \times \tilde{\vec{H}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{2\gamma Z_o[-\tilde{\vec{E}}(s) \cdot \tilde{\vec{H}}(-s) - \tilde{\vec{H}}(s) \cdot \tilde{\vec{E}}(-s)]$$

$$+ [\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}_m(-s) + Z_o^2\tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}(-s)]$$

$$+ [-\tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}_m(s) - Z_o^2\tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}(s)]\Big\}dV \qquad (11.4)$$

The opposite frequency energy theorem, from equation 9.3, is

$$\int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{H}}(-s) + \tilde{\vec{H}}(s) \times \tilde{\vec{E}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{2\frac{\gamma}{Z_o}[\tilde{\vec{E}}(s) \cdot \tilde{\vec{E}}(-s) - Z_o^2\tilde{\vec{H}}(s) \cdot \tilde{\vec{H}}(-s)]$$

$$+ [-\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}(-s) + \tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}_m(-s)]$$

$$+ [\tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}(s) - \tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}_m(s)]\Big\}dV \qquad (11.5)$$

B. Reciprocity for opposite frequencies: the case of $q_1 = -q_2 \equiv q$

The opposite frequencies combined reciprocity theorem, equation 9.4, becomes

$$\int_S [\tilde{\vec{F}}_q(s) \times \tilde{\vec{F}}_{-q}(-s)] \cdot \vec{n}dS$$

$$= qiZ_o \int_V [\tilde{\vec{F}}_q(s) \cdot \tilde{\vec{K}}_{-q}(-s) + \tilde{\vec{F}}_{-q}(-s) \cdot \tilde{\vec{K}}_q(s)]dV \quad (11.6)$$

The opposite frequencies magnetic reciprocity theorem, from equation 9.5, is

$$\int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{E}}(-s) + Z_o^2 \tilde{\vec{H}}(s) \times \tilde{\vec{H}}(-s)] \cdot \vec{n}dS$$

$$= \int_V [\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}_m(-s) - Z_o^2 \tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}(-s)$$

$$-\tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}_m(s) + Z_o^2 \tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}(s)]dV \quad (11.7)$$

The opposite frequencies electric reciprocity theorem, from equation 9.6, is

$$\int_S [-\tilde{\vec{E}}(s) \times \tilde{\vec{H}}(-s) + \tilde{\vec{H}}(s) \times \tilde{\vec{E}}(-s)] \cdot \vec{n}dS$$

$$= \int_V [\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}(-s) + \tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}_m(-s)$$

$$+ \tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}(s) + \tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}_m(s)]dV \quad (11.8)$$

C. Combinations of previous results (including generalized Poynting vector theorem)

Equation 9.7 becomes

$$\int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{E}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \Big\{ \gamma Z_o [-\tilde{\vec{E}}(s) \cdot \tilde{\vec{H}}(-s) - \tilde{\vec{H}}(s) \cdot \tilde{\vec{E}}(-s)]$$

$$+ [\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}_m(-s) - \tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}_m(s)] \Big\} dV \quad (11.9)$$

Equation 9.8 becomes

$$\int_S [\tilde{\vec{H}}(s) \times \tilde{\vec{H}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}(s) \cdot \tilde{\vec{H}}(-s) + \tilde{\vec{H}}(s) \cdot \tilde{\vec{E}}(-s)] \right.$$

$$\left. + [-\tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}(-s) + \tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}(s)] \right\} dV \qquad (11.10)$$

Equation 9.9 becomes

$$\int_S [\tilde{\vec{E}}(s) \times \tilde{\vec{H}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}(s) \cdot \tilde{\vec{E}}(-s) - Z_o^2\tilde{\vec{H}}(s) \cdot \tilde{\vec{H}}(-s)] \right.$$

$$\left. + [-\tilde{\vec{E}}(s) \cdot \tilde{\vec{J}}(-s) - \tilde{\vec{H}}(-s) \cdot \tilde{\vec{J}}_m(s)] \right\} dV \qquad (11.11)$$

Equation 9.10 becomes

$$\int_S [\tilde{\vec{H}}(s) \times \tilde{\vec{E}}(-s)] \cdot \vec{n}dS$$

$$= \int_V \left\{ \frac{\gamma}{Z_o} [\tilde{\vec{E}}(s) \cdot \tilde{\vec{E}}(-s) - Z_o^2\tilde{\vec{H}}(s) \cdot \tilde{\vec{H}}(-s)] \right.$$

$$\left. + [\tilde{\vec{H}}(s) \cdot \tilde{\vec{J}}_m(-s) + \tilde{\vec{E}}(-s) \cdot \tilde{\vec{J}}(s)] \right\} dV \qquad (11.12)$$

Equation 11.11 is the usual Poynting vector theorem extended to the complex frequency domain with magnetic current density also included. Comparing with the usual form of the Poynting vector theorem note the use of complex conjugation in the usual form.[9] The form in equation 11.11 avoids this problem and is analytic with respect to the complex frequency. Note that equation 9.10 is an alternate form of the Poynting vector theorem since equations 11.11 and 11.12 are equivalent; this is readily seen by substituting -s for s (and conversely) in equation 11.11. Thus we have two forms for our generalized Poynting vector theorem.

Either of them or a linear combination of the two (as in equations 11.5 and 11.8) can be used depending on convenience.

D. The case of $s = 0$

For zero complex frequency the terms involving dot products of the electric and magnetic fields with other electric and/or magnetic field terms become zero. The remaining terms involve dot products of fields with current densities. The equations in part A of this section (equations 11.3, 11.4, and 11.5) become trivial. In part B equation 11.7 becomes trivial. In part C equations 11.9 and 11.10 become trivial. The generalized Poynting vector theorems reduce to the same thing and become identical to equation 6.4 for $s = 0$.

XII. Green's Functions for Retarded and Advanced Fields Including the Far Field Forms

A. Green's functions and fields

As a prelude to letting the surface S tend to infinity in the previous results consider the Green's function for calculating the fields from the current density. As discussed in a previous note[2] we have the free space Green's functions as

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma R}}{4\pi R}$$

$$\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = \vec{e}_R \frac{1}{4\pi} [-R^{-2} - \gamma R^{-1}]\, e^{-\gamma R}$$

$$\tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';s) = \left[\vec{\vec{I}} - \frac{1}{\gamma^2}\nabla\nabla\right]\tilde{G}_o(\vec{r},\vec{r}\,';s) \tag{12.1}$$

$$= \frac{\gamma}{4\pi}\left\{[(\gamma R)^{-3} + (\gamma R)^{-2} + (\gamma R)^{-1}]\, e^{-\gamma R}\vec{\vec{I}} + [-3(\gamma R)^{-3} - 3(\gamma R)^{-2} - (\gamma R)^{-1}]\, e^{-\gamma R}\vec{e}_R\vec{e}_R\right\}$$

where

$$\vec{\vec{I}} \equiv \vec{e}_R\vec{e}_R + \vec{e}_\Theta\vec{e}_\Theta + \vec{e}_\Phi\vec{e}_\Phi$$

$$R \equiv |\vec{r} - \vec{r}\,'| \tag{12.2}$$

$$\vec{e}_R \equiv \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|}$$

and (R, Θ, Φ) form a spherical coordinate system centered on $\vec{r}\,'$. These are the retarded Green's functions for free space. The corresponding advanced Green's functions are

$$\tilde{G}_a(\vec{r},\vec{r}\,';s) = \frac{e^{\gamma R}}{4\pi R}$$

$$\nabla\tilde{G}_a(\vec{r},\vec{r}\,';s) = \frac{1}{4\pi}[-R^{-2} + \gamma R^{-1}]\ e^{\gamma R}$$

$$\vec{\vec{G}}_a(\vec{r},\vec{r}\,';s) = [\vec{\vec{I}} - \nabla\nabla]\tilde{G}_a(\vec{r},\vec{r}\,';s) \tag{12.3}$$

$$= \frac{\gamma}{4\pi}\Big\{[(\gamma R)^{-3} - (\gamma R)^{-2} + (\gamma R)^{-1}]\ e^{\gamma R}\vec{\vec{I}}$$

$$+ [-3(\gamma R)^{-3} + 3(\gamma R)^{-2} - (\gamma R)^{-1}]\ e^{\gamma R}\vec{e}_R\vec{e}_R\Big\}$$

Having expressions for the advanced and retarded Green's functions note that they are related by a frequency reversal as

$$\tilde{G}_a(\vec{r},\vec{r}\,';s) = \tilde{G}_o(\vec{r},\vec{r}\,';-s)$$

$$\nabla\tilde{G}_a(\vec{r},\vec{r}\,';s) = \nabla\tilde{G}_o(\vec{r},\vec{r}\,';-s) \tag{12.4}$$

$$\tilde{\vec{\vec{G}}}_a(\vec{r},\vec{r}\,';s) = \tilde{\vec{\vec{G}}}_o(\vec{r},\vec{r}\,';-s)$$

The free space Green's functions (retarded) are used for outgoing waves. From the point $\vec{r}\,'$ they travel with time argument $t - |\vec{r} - \vec{r}\,'|/c$. The corresponding advanced Green's functions are used for incoming waves. To the point $\vec{r}\,'$ they travel with time argument $t + |\vec{r} - \vec{r}\,'|/c$.

The generalized form of Maxwell's equations in free space (with two frequencies s and s') presented in section VII (the first of equations 7.5 in particular) has retarded and advanced solutions as

$$\tilde{\vec{F}}_{r_q}(\vec{r},s) = -s\mu_o\langle\tilde{\vec{\vec{G}}}_o(s);\tilde{\vec{K}}_q(s')\rangle + qiZ_o\langle\nabla\tilde{G}_o(s)\overset{\times}{;}\tilde{\vec{K}}_q(s')\rangle$$

$$\tilde{\vec{F}}_{a_q}(\vec{r},s) = -s\mu_o\langle\tilde{\vec{\vec{G}}}_a(s);\tilde{\vec{K}}_q(s')\rangle + qiZ_o\langle\nabla\tilde{G}_a(s)\overset{\times}{;}\tilde{\vec{K}}_q(s')\rangle \tag{12.5}$$

where the symmetric product notation is as introduced previously.[4] Restricted linear combinations of these results also solve the generalized Maxwell equations provided the matrix coefficients are adjusted by associating a part of each of the current densities with each of the two solution types.

B. Far Green's functions and far fields

For large r with bounded $\vec{r}\,'$ we have

$$|\vec{r} - \vec{r}\,'| = r\left[1 - 2\frac{\vec{e}_r \cdot \vec{r}\,'}{r} + \frac{r'^2}{r^2}\right]^{1/2}$$

$$= r - \vec{e}_r \cdot \vec{r}\,' + O(r^{-1})$$

$$|\vec{r} - \vec{r}\,'|^{-1} = \frac{1}{r} + O(r^{-2}) \tag{12.6}$$

$$\vec{e}_R = \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|} = \left[\frac{\vec{r}}{r} - \frac{\vec{r}\,'}{r}\right][1 + O(r^{-1})]$$

$$= \vec{e}_r + O(r^{-1})$$

This carries over for the retarded Green's functions as

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma r+\gamma\vec{e}_r\cdot\vec{r}\,'+O(r^{-1})}}{4\pi r[1 + O(r^{-1})]}$$

$$= \tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{-\gamma r}}{r}\,[1 + O(r^{-1})]$$

$$\tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s) \equiv \frac{e^{\gamma\vec{e}_r\cdot\vec{r}\,'}}{4\pi}$$

$$\nabla\tilde{G}_o(\vec{r},\vec{r}\,';s) = [\vec{e}_r + O(r^{-1})]\,\frac{e^{-\gamma r+\gamma\vec{e}_r\cdot\vec{r}\,'+O(r^{-1})}}{4\pi}[-\gamma r + O(r^{-2})]$$

$$= -\vec{e}_r\gamma\tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{-\gamma r}}{r}\,[1 + O(r^{-1})] \tag{12.7}$$

$$\tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma r+\gamma\vec{e}_r\cdot\vec{r}\,'+O(r^{-1})}}{4\pi r}[\overleftrightarrow{I} - \vec{e}_R\vec{e}_R][1 + O(r^{-1})]$$

$$= \tilde{\overleftrightarrow{g}}_{o_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{-\gamma r}}{r}[1 + O(r^{-1})]$$

$$\tilde{\overleftrightarrow{g}}_{o_f}(\vec{e}_r,\vec{r}\,';s) \equiv \tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';s)\overleftrightarrow{T}$$

$$\overleftrightarrow{T} \equiv \overleftrightarrow{I} - \vec{e}_r\vec{e}_r$$

$$\overleftrightarrow{T}\cdot = -\vec{e}_r \times \vec{e}_r \times$$

Note the use of the identity dyadic $\overleftrightarrow{I}$, the transverse dyadic $\overleftrightarrow{T}$, and the far Green's functions $\tilde{g}_{of}$ and $\tilde{\overleftrightarrow{g}}_{of}$ for retarded fields. The advanced Green's functions for large r are

$$\tilde{G}_a(\vec{r},\vec{r}\,';s) = \tilde{g}_{a_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{\gamma r}}{r}[1 + O(r^{-1})]$$

$$\tilde{g}_{a_f}(\vec{e}_r,\vec{r}\,';s) \equiv \frac{e^{-\gamma\vec{e}_r\cdot\vec{r}\,'}}{4\pi}$$

$$\nabla\tilde{G}_a(\vec{r},\vec{r}\,';s) = \vec{e}_r\gamma\tilde{g}_{a_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{\gamma r}}{r}[1 + O(r^{-1})] \tag{12.8}$$

$$\tilde{\overleftrightarrow{G}}_a(\vec{r},\vec{r}\,';s) = \tilde{\overleftrightarrow{g}}_{a_f}(\vec{e}_r,\vec{r}\,';s)\,\frac{e^{\gamma r}}{r}[1 + O(r^{-1})]$$

$$\tilde{\overleftrightarrow{g}}_{a_f}(\vec{e}_r,\vec{r}\,';s) \equiv \tilde{g}_{a_f}(\vec{e}_r,\vec{r}\,';s)\overleftrightarrow{T}$$

The far Green's functions for advanced and retarded fields are related by

$$\tilde{g}_{a_f}(\vec{e}_r,\vec{r}\,';s) = \tilde{g}_{o_f}(\vec{e}_r,\vec{r}\,';-s)$$

$$\tilde{\overleftrightarrow{g}}_{a_f}(\vec{e}_r,\vec{r}\,';s) = \tilde{\overleftrightarrow{g}}_{o_f}(\vec{e}_r,\vec{r}\,';-s) \qquad (12.9)$$

The far fields are[2]

$$r\tilde{\vec{F}}_{r_{f_q}}(\vec{r},s) \equiv \lim_{r\to\infty} r\, e^{\gamma r}\tilde{\vec{F}}_{r_q}(\vec{r},s)$$

$$r\tilde{\vec{F}}_{a_{f_q}}(\vec{r},s) \equiv \lim_{r\to\infty} r\, e^{-\gamma r}\tilde{\vec{F}}_{a_q}(\vec{r},s) \qquad (12.10)$$

Equations 12.5 for the far fields become

$$r\tilde{\vec{F}}_{r_{f_q}}(\vec{r},s) = -s\mu_o[\overleftrightarrow{I} - \vec{e}_r\vec{e}_r + qi\vec{e}_r \times \overleftrightarrow{I}] \cdot \left\langle \tilde{g}_{o_f}(s), \tilde{\vec{K}}_q(s') \right\rangle$$

$$r\tilde{\vec{F}}_{a_{f_q}}(\vec{r},s) = -s\mu_o[\overleftrightarrow{I} - \vec{e}_r\vec{e}_r - qi\vec{e}_r \times \overleftrightarrow{I}] \cdot \left\langle \tilde{g}_{a_f}(s), \tilde{\vec{K}}_q(s') \right\rangle \qquad (12.11)$$

Note that if we interchange s and -s while keeping s' fixed the roles of advanced and retarded solutions are not in general interchanged. While such a switch does interchange the retarded and advanced Green's functions one would need to change the sign of q in the coefficients on the right hand side. However, this would also change the sign of the q subscript on the currents which would make the right hand side retarded form not go to exactly the corresponding advanced form. If the magnetic current density (or alternatively the electric current density) were set to zero such a transformation could be made. This can be seen in both equations 12.5 and 12.11.

C. Some vector and matrix identities

For use with the far field results of this section we have some vector and matrix identities. Letting $\vec{\alpha}$ and $\vec{\beta}$ be two arbitrary 3 component vectors we have first

$$(\overleftrightarrow{T} \cdot \vec{\alpha}) \times (\overleftrightarrow{T} \cdot \vec{\beta}) \equiv [(\overleftrightarrow{I} - \vec{e}_r\vec{e}_r) \cdot \vec{\alpha}] \times [(\overleftrightarrow{I} - \vec{e}_r\vec{e}_r) \cdot \vec{\beta}]$$

$$= [\vec{e}_r \times (\vec{e}_r \times \vec{\alpha})] \times [\vec{e}_r \times (\vec{e}_r \times \vec{\beta})]$$

$$= \vec{\alpha}_T \times \vec{\beta}_T$$

$$= \vec{e}_r\vec{e}_r \cdot (\vec{\alpha} \times \vec{\beta}) \qquad (12.12)$$

$$(\vec{e}_r \times \vec{\alpha}) \times (\vec{e}_r \times \vec{\beta})$$

$$= \vec{e}_r\vec{e}_r \cdot (\vec{\alpha} \times \vec{\beta})$$

so that

$$(\overleftrightarrow{T} \cdot \vec{\alpha}) \times (\overleftrightarrow{T} \cdot \vec{\beta})$$

$$= (\vec{e}_r \times \vec{\alpha}) \times (\vec{e}_r \times \vec{\alpha})$$

$$= \vec{\alpha}_T \times \vec{\beta}_T$$

$$= \vec{e}_r\vec{e}_r \cdot (\vec{\alpha} \times \vec{\beta}) \qquad (12.13)$$

where the subscript T indicates the transverse part with respect to $\vec{e}_r$, i.e.

$$\vec{\alpha}_T \equiv \overleftrightarrow{T} \cdot \vec{\alpha} = -\vec{e}_r \times (\vec{e}_r \times \vec{\alpha}) \qquad (12.14)$$

Similarly we have

$$(\vec{\vec{T}} \cdot \vec{\alpha}) \times (\vec{e}_r \times \vec{\beta})$$

$$= (\vec{\vec{T}} \cdot \vec{\beta}) \times (\vec{e}_r \times \vec{\alpha})$$

$$= \vec{e}_r\vec{e}_r \cdot [\vec{\alpha} \times (\vec{e}_r \times \vec{\beta})]$$

$$= \vec{e}_r\vec{e}_r \cdot [\vec{\beta} \times (\vec{e}_r \times \vec{\alpha})]$$

$$= \vec{e}_r[(\vec{\vec{T}} \cdot \vec{\alpha}) \cdot (\vec{\vec{T}} \cdot \vec{\beta})]$$

$$= \vec{e}_r(\vec{\alpha}_T \cdot \vec{\beta}_T) \tag{12.15}$$

These identities can be readily derived from the more common ones.[7]

XIII. Infinite Boundary Surface

In previous sections we have considered the case of a volume V bounded by a closed surface S where S had finite linear dimensions. Now consider a volume V_o which contains all the current (and charge) densities. Let V_o be contained within V and let V be a simply connected region so that we can let $S \to \infty$, denoted by S_∞, and V can be considered V_∞, i.e. all space. In letting $S \to \infty$ let us for convenience consider S to be a sphere of radius r centered on the coordinate origin. Furthermore let S' denote the unit sphere where integrals are taken over θ, ϕ at infinity without r present in the integral. For these choices we have

$$dS = r^2 \sin(\theta) d\theta d\phi$$

$$dS' = \sin(\theta) d\theta d\phi \qquad (13.1)$$

$$\int_{S'} dS' = 4\pi$$

Note that the symmetric products of Green's functions as introduced in the last section are simply integrals over V_o with respect to the $\vec{r}\,'$ coordinates.

A. Some considerations on convergence of the surface integral over S_∞

From the far field expressions (equations 12.1) and the order of the accuracy of these expressions as indicated in the large r expansions of the Green's functions (equations 12.7 and 12.8) one can make a first categorization of the reciprocity/energy theorems based on the behavior of the terms in the surface integral as $r \to \infty$. Let us start from the most general form, the mixed frequencies combined reciprocity and energy theorem (equation 7.6). For S_∞ ($S \to \infty$) the most important factors are in the far field expressions, since they are exponentials in frequency times radius. The current densities are confined to V_o with finite radius. For the current densities the dependence on s_1' and s_2' is not significant in comparison to the field dependence on s_1 and s_2 as $r \to \infty$. Let us then consider the field dependence in the surface integral for various s_1 and s_2 as $r \to \infty$.

Define

$$I \equiv \int_S [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n} dS$$

$$= \int_V [(q_1 i\gamma_1 - q_2 i\gamma_2)\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{F}}^{(2)}_{q_2}(s_2)$$

$$+ q_1 i Z_o \tilde{\vec{F}}^{(2)}_{q_2}(s_2) \cdot \tilde{\vec{K}}^{(1)}_{q_1}(s_1') - q_2 i Z_o \tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{K}}^{(2)}_{q_2}(s_2')] dV \tag{13.2}$$

Constrain now that S contain V_o so that all the current densities are inside of S. This is convenient in that for some cases I is then independent of S and is the same for such cases as its value for S_∞.

Then where the limit exists define

$$I_\infty \equiv \lim_{r\to\infty} I$$

$$= \lim_{r\to\infty} \int_S [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n} dS$$

$$= \int_{S_\infty} [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n} dS$$

$$= \left[\lim_{r\to\infty} e^{\mp\gamma_1 r \mp \gamma_2 r}\right] \int_{S'} [[r\tilde{\vec{F}}_{r,a_{f_{q_1}}}(\vec{r},s_1)]$$

$$\times [r\tilde{\vec{F}}_{r,a_{f_{q_2}}}(\vec{r},s_2)]] \cdot \vec{e}_r dS' \tag{13.3}$$

The neglected term in I_∞ is $O(r^{-1})$ times the same exponentials that give I_∞. Thus we consider the effect of s_1 and s_2 is determining I_∞. The effect of choosing retarded and/or advanced solutions is indicated by the - or + and r, a respectively in the last part of equation 13.3. Thus define

$$I_f \equiv \int_{S'} \left[\left[r\tilde{\vec{F}}_{r,a_{f_{q_1}}}(\vec{r},s_1)\right] \times \left[r\tilde{\vec{F}}_{r,a_{f_{q_2}}}(\vec{r},s_2)\right]\right] \cdot \vec{e}_r dS' \tag{13.4}$$

$$I_\infty = \left[\lim_{r\to\infty} e^{\mp\gamma_1 r \mp \gamma_2 r}\right] I_f$$

This leads to the following table concerning choices of s_1 and s_2 provided I_f is finite and where $\delta > 0$ is some small number.

Solution 1	Solution 2	$I_\infty = 0$	$I_\infty = I_f$	I_∞ infinite unless special conditions
retarded	retarded	$Re[s_1 + s_1] \geq \delta > 0$	$Re[s_1 + s_2] = 0$	$Re[-s_1 - s_2] \geq \delta > 0$
retarded	advanced	$Re[s_1 - s_2] \geq \delta > 0$	$Re[s_1 - s_2] = 0$	$Re[-s_1 + s_2] \geq \delta > 0$
advanced	retarded	$Re[-s_1 + s_2] \geq \delta > 0$	$Re[s_1 - s_2] = 0$	$Re[s_1 - s_2] \geq \delta > 0$
advanced	advanced	$Re[-s_1 - s_2] \geq \delta > 0$	$Re[s_1 + s_2] = 0$	$Re[s_1 + s_2] \geq \delta > 0$

Table 13.1 Implications of combinations of frequencies with retarded and/or advanced solutions on integral convergence

The special conditions referred to in the right hand column include at a minimum that $I_f = 0$. Even this is not in general sufficient since it involves only the far field (r^{-1}) terms. The higher order negative powers in r are not sufficient by themselves to dominate a growing exponential. In some cases $I \equiv 0$ as can be demonstrated by analytic continuation from the case $I_\infty = 0$ provided the volume integral is only over V_o, the volume of finite dimensions containing the current densities.

B. Choices for combinations of the various frequencies

In choosing useful special cases for consideration of the surface integral (and its equivalent volume integral I as $r \to \infty$), let us look for special features of the integrand of the volume integral. Write I as

$$I \equiv \int_S [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n}dS$$

$$= \int_V (q_1 i\gamma_1 - q_2 i\gamma_2)\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{F}}^{(2)}_{q_2}(s_2)dV$$

$$+ \int_{V_o} [q_1 iZ_o \tilde{\vec{F}}^{(2)}_{q_2}(s_2) \cdot \tilde{\vec{K}}^{(1)}_{q_1}(s_1') - q_2 iZ_o \tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{K}}^{(2)}_{q_2}(s_2')]dV \tag{13.5}$$

Since the current densities are assumed confined to V_o (with finite dimensions) then only the volume integral involving dot products of two fields can affect the limit as $r \to \infty$. Then if we set

$$q_1\gamma_1 = q_2\gamma_2 \tag{13.6}$$

the integral over V is zero. This has two cases as

$$q_1 = q_2 \equiv q \ , \ s_1 = s_2 \equiv s \quad \text{or} \quad q_1 = -q_2 \equiv \gamma \ , \ s_1 = -s_2 \equiv s \tag{13.7}$$

For such cases we have

$$I = I_\infty$$

$$= \int_{S_\infty} [\tilde{\vec{F}}^{(1)}_{q_1}(s_1) \times \tilde{\vec{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n}dS$$

$$= \int_{V_o} [q_1 iZ_o \tilde{\vec{F}}^{(2)}_{q_2}(s_2) \cdot \tilde{\vec{K}}^{(1)}_{q_1}(s_1') - q_2 iZ_o \tilde{\vec{F}}^{(1)}_{q_1}(s_1) \cdot \tilde{\vec{K}}^{(2)}_{q_2}(s_2')]dV \tag{13.8}$$

Note that the choice of retarded and/or advanced fields does not change the form of the above results or their convergence as $r \to \infty$. This set of choices as in equations 13.7 gives what we have called reciprocity theorems.

A complementary choice to equation 13.6 is

$$q_1\gamma_1 = -q_2\gamma_2 \tag{13.9}$$

which has two cases as

$$q_1 = q_2 \equiv q \ , \ s_1 = -s_2 \equiv s \quad \text{or} \quad q_1 = -q_2 \equiv q \ , \ s_1 = s_2 \equiv s \tag{13.10}$$

For such cases we have (for which cases it exists)

$$\begin{aligned} I &= I_\infty \\ &= \int_{S_\infty} [\vec{\tilde{F}}^{(1)}_{q_1}(s_1) \times \vec{\tilde{F}}^{(2)}_{q_2}(s_2)] \cdot \vec{n}\,dS \\ &= qi2\gamma \int_{V_\infty} \vec{\tilde{F}}^{(1)}_{q_1}(s_1) \cdot \vec{\tilde{F}}^{(2)}_{q_2}(s_2)\,dV \\ &\quad + \int_{V_o} [q_1 iZ_o \vec{\tilde{F}}^{(2)}_{q_2}(s_2) \cdot \vec{\tilde{K}}^{(1)}_{q_1}(s_1') - q_2 iZ_o \vec{\tilde{F}}^{(1)}_{q_1}(s_1) \cdot \vec{\tilde{K}}^{(2)}_{q_2}(s_2')]\,dV \end{aligned} \tag{13.11}$$

In this result the choice of retarded and/or advanced fields does affect the convergence of the integral over V_∞. This set of choices as in equations 13.10 gives what we have called energy theorems.

Note that the results of this section have not depended on the choice of s_1' and s_2'. Later we will require $s_1' = s_1$ and $s_2' = s_2$ for special results.

XIV. Reaction and Subreaction

In the various equations that result from special choices for the variables in equation 13.5 (as in sections III, IV, VIII, and IX) there are various combinations (dot products integrated over the volume containing the current densities) of fields and current densities. One such combination (equation 1.2) has been called the reaction.[12] The present note shows that there are several similar combinations of fields and current densities with similar roles to play.

Let us then define a few terms. By reaction I mean the linear combination of symmetric products of fields with current densities such as appear in these equations for the case that the symmetric products of fields with fields are identically zero. This latter condition defines the reciprocity equations. For cases that the symmetric products of fields with fields are non zero the symmetric products of fields with currents can be referred to as subreaction. Subreaction is then associated with the energy equations.

Denote the reaction between the 1 fields and 2 current densities by $R(1,s_1;2,s_2)$ where the two possibly different frequencies s_1 and s_2 are exhibited. Then we have the combined reaction for identical frequencies from equation 4.1 as

$$R_{c_q}(1,s;2,s) \equiv \left\langle \tilde{\vec{F}}_q^{(1)}(s);\tilde{\vec{K}}_q^{(2)}(s)\right\rangle$$

$$= R_e(1,s;2,s) + qiR_m(1,s;2,s) \qquad (14.1)$$

Note that the symmetric products for reaction (and subreaction) are integrals over V_o, the symmetric products involving two fields are integrals over V_∞. The electric reaction for identical frequencies from equation 4.3 is

$$R_e(1,s;2,s) = \left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(s)\right\rangle - \left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(s)\right\rangle \qquad (14.2)$$

The magnetic reaction for identical frequencies from equation 4.2 is

$$R_m(1,s;2,s) = \frac{1}{Z_o}\left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(s)\right\rangle + Z_o\left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(s)\right\rangle \qquad (14.3)$$

For opposite frequencies we can similarly define a combined reaction from equation 9.4 as

$$R_{c_q}(1,s;2,-s) \equiv \left\langle \tilde{\vec{F}}_q^{(1)}(s);\tilde{\vec{K}}_{-q}^{(2)}(-s)\right\rangle$$

$$= R_e(1,s;2,-s) + qiR_m(1,s;2,-s) \tag{14.4}$$

The electric reaction for opposite frequencies from equation 9.6 is

$$R_e(1,s;2,-s) = \left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(-s)\right\rangle + \left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(-s)\right\rangle \tag{14.5}$$

The magnetic reaction for opposite frequencies from equation 9.5 is

$$R_m(1,s;2,-s) = -\frac{1}{Z_o}\left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(-s)\right\rangle + Z_o\left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(-s)\right\rangle \tag{14.6}$$

Next denote the subreaction between the 1 fields and the 2 current densities by $R'(1,s_1;2,s_2)$. The combined subreaction for identical frequencies from equation 4.4 is

$$R'_{c_q}(1,s;2,s) \equiv \left\langle \tilde{\vec{F}}_q^{(1)}(s);\tilde{\vec{K}}_{-q}^{(2)}(s)\right\rangle$$

$$= R'_e(1,s;2,s) + qiR'_m(1,s;2,s) \tag{14.7}$$

The electric subreaction for identical frequencies from equation 4.6 is

$$R'_e(1,s;2,s) = \left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(s)\right\rangle + \left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(s)\right\rangle \tag{14.8}$$

The magnetic subreaction for identical frequencies from equation 4.5 is

$$R'_m(1,s;2,s) = -\frac{1}{Z_o}\left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(s)\right\rangle + Z_o\left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(s)\right\rangle \tag{14.9}$$

For opposite frequencies a combined subreaction can be defined from equation 9.1 as

$$R'_{c_q}(1,s;2,-s) \equiv \left\langle \tilde{\vec{F}}_q^{(1)}(s);\tilde{\vec{K}}_q^{(2)}(-s) \right\rangle$$

$$= R'_e(1,s;2,-s) + qiR'_m(1,s;2,-s) \quad (14.10)$$

The electric subreaction for opposite frequencies from equation 9.3 is

$$R'_e(1,s;2,-s) = \left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(-s) \right\rangle - \left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(-s) \right\rangle \quad (14.11)$$

The magnetic subreaction for opposite frequencies from equation 9.2 is

$$R'_m(1,s;2,-s) = \frac{1}{Z_o}\left\langle \tilde{\vec{E}}^{(1)}(s);\tilde{\vec{J}}_m^{(2)}(-s) \right\rangle + Z_o\left\langle \tilde{\vec{H}}^{(1)}(s);\tilde{\vec{J}}^{(2)}(-s) \right\rangle \quad (14.12)$$

For convenience let d = +1 or d = r be used to symbolize a retarded solution and d = -1 or d = a be used to symbolize an advanced solution. Then a reaction such as in equation 14.1 can be written as

$$R_{c_q}(1,d_1,s;2,d_2,s) = \left\langle \tilde{\vec{F}}_{d_{1_q}}^{(1)}(s);\tilde{\vec{K}}_{d_{2_q}}^{(2)}(s) \right\rangle \quad (14.13)$$

where d_1 and d_2 can indicate any combination of retarded and advanced. This notation can readily be applied to all the reactions and subreactions in this section.

A related quantity is the integral of the far field quantities over S'. Equations 13.4 can now be written as

$$I_f(1,d_1,q_1,s_1;2,d_2,q_2,s_2)$$

$$\equiv \int_{S'} \left[[r\tilde{\vec{F}}_{d_{1_{f_{q_1}}}}^{(1)}(\vec{r},s_1)] \times [r\tilde{\vec{F}}_{d_{2_{f_{q_2}}}}^{(2)}(\vec{r},s_2)] \right] \cdot \vec{e}_r dS' \quad (14.14)$$

$$I_\infty = \left[\lim_{r\to\infty} e^{-d_1\gamma_1 r - d_2\gamma_2 r} \right] I_f$$

XV. Combined Reciprocity and Energy with Identical Frequencies for Infinite Boundary Surface

Initially choose

$$s_1 = s_2 \equiv s \; , \qquad s_1' = s_2' \equiv s' \tag{15.1}$$

Then consider the surface integral I_f as defined in section XIII. This will lead to some general results for the reaction and subreaction quantities and for the surface integral I for general S (remembering that S contains V_o).

A. Reciprocity: the case of $q_1 = q_2 \equiv q$

Equation 13.8 applies to this case in the form

$$I = I_\infty = \int_{S_\infty} [\tilde{\vec{F}}_q^{(1)}(s) \times \tilde{\vec{F}}_q^{(2)}(s)] \cdot \vec{n} dS$$

$$= qiZ_o [\langle \tilde{\vec{F}}_q^{(2)}(s); \tilde{\vec{K}}_q^{(1)}(s') \rangle - \langle \tilde{\vec{F}}_q^{(1)}(s); \tilde{\vec{K}}_q^{(2)}(s') \rangle]$$

$$= qiZ_o [R_{c_q}(2,s;1,s') - R_{c_q}(1,s;2,s')] \tag{15.2}$$

For both solutions retarded $I_\infty = 0$ for $Re[s] \geq \delta > 0$; the integral over V_o is then zero for such s and by analytic continuation throughout the s plane. For both solutions advanced $I_\infty = 0$ for $Re[s] \leq \delta < 0$; the integral over V_o is then zero for such s and by analytic continuation throughout the s plane. Then for either both solutions retarded or both solutions advanced we have

$$I = I_\infty = \left[\lim_{r\to\infty} e^{-2d\gamma r}\right] I_f$$

$$I_f(1,d,q,s;2,d,q,s) = \int_{S'} [[r\tilde{\vec{F}}_{d_{f_q}}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{F}}_{d_{r_q}}^{(2)}(\vec{r},s)]] \cdot \vec{e}_r dS$$

$$= 0$$

$$= -I_f(2,d,q,s;1,d,q,s) \tag{15.3}$$

Consider now that one solution is retarded and the other advanced. Then we have from equations 12.11 through 12.15 and equations 13.4 the result

$$I = I_\infty = I_f(1,d,q,s;2,-d,q,s)$$

$$= \int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_{f_q}}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_{f_q}}(\vec{r},s)]] \cdot \vec{e}_r dS'$$

$$= 2(s\mu_o)^2 \int_{S'} \Big\{ [\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}^{(1)}_q(s')\rangle$$

$$\times \langle \tilde{g}_{-d_f}(s), \tilde{\vec{K}}^{(2)}_q(s')\rangle] \cdot \vec{e}_r$$

$$-qdi \langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}^{(1)}_q(s')\rangle_T \cdot \langle \tilde{g}_{-d_f}(s), \tilde{\vec{K}}^{(2)}_q(s')\rangle_T \Big\} dS'$$

$$= -I_f(2,-d,q,s;1,d,q,s) \tag{15.4}$$

Note that for present purposes

$$\tilde{g}_{r_f} \equiv \tilde{g}_{o_f} \equiv \tilde{g}_{1_f}$$

$$\tilde{g}_{a_f} \equiv \tilde{g}_{-1_f} \tag{15.5}$$

The results of equations 15.3 and 15.4 apply throughout the s plane for the appropriate retarded/advanced combinations for the fields. Note that these results are not dependent on our choice of s', the frequency variable for the current densities. Thus s' can be chosen as some complex frequency, the above results as in equation 15.3 derived with s in the appropriate half plane for convergence, and then s can be changed to s' by analytic continuation. In particular then the above results are valid for

$$s \equiv s' \tag{15.6}$$

which we now choose. Of course the terms must be appropriately bounded such that the integrals exist, and this may restrict our choice of s in some cases.

Consider then the case that both solutions are retarded or both advanced. For these cases since I = 0 we have the combined reaction theorem from equation 15.2 as

$$R_{c_q}(1,d,s;2,d,s) = R_{c_q}(2,d,s;1,d,s) \tag{15.7}$$

Split this to give the electric reaction theorem

$$R_e(1,d,s;2,d,s) = R_e(2,d,s;1,d,s) \tag{15.8}$$

and the magnetic reaction theorem

$$R_m(1,d,s;2,d,s) = R_m(2,d,s;1,d,s) \tag{15.9}$$

For this case the surface integral (S enclosing V_o) is zero giving in combined form

$$\begin{aligned} &\int_S [\tilde{\vec{F}}^{(1)}_{d_q}(s) \times \tilde{\vec{F}}^{(2)}_{d_q}(s)] \cdot \vec{n} = 0 \\ &\int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_{f_q}}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{d_{f_q}}(\vec{r},s)]] \cdot \vec{e}_r = 0 \end{aligned} \tag{15.10}$$

which can be split to give

$$\begin{aligned} &\int_S [\tilde{\vec{E}}^{(1)}_d(s) \times \tilde{\vec{E}}^{(2)}_d(s) - Z_o^2 \tilde{\vec{H}}^{(1)}_d(s) \times \tilde{\vec{H}}^{(2)}_d(s)] \cdot \vec{n} = 0 \\ &\int_{S'} [[r\tilde{\vec{E}}^{(1)}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{E}}^{(2)}_{d_f}(\vec{r},s)] \\ &\qquad - Z_o^2 (r\tilde{\vec{H}}^{(1)}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{H}}^{(2)}_{d_f}(\vec{r},s)]] \cdot \vec{e}_r = 0 \end{aligned} \tag{15.11}$$

and

$$\int_S [\tilde{\vec{E}}_d^{(1)}(s) \times \tilde{\vec{H}}_d^{(2)}(s) + \tilde{\vec{H}}_d^{(1)}(s) \times \tilde{\vec{E}}_d^{(2)}(s)] \cdot \vec{n} dS = 0$$

$$\int_{S'} [[r\tilde{\vec{E}}_{d_f}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{H}}_{d_f}^{(2)}(\vec{r},s)] \tag{15.12}$$

$$+ [r\tilde{\vec{H}}_{d_f}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{E}}_{d_f}^{(2)}(\vec{r},s)]] \cdot \vec{e}_r = 0$$

The case that one solution is retarded and the other advanced does not give as simple results. For mixed retarded and advanced solutions we have

$$R_{c_q}(1,d,s;2,-d,s) - R_{c_q}(2,-d,s;1,d,s)$$

$$= q \frac{i}{Z_o} I_f(1,d,q,s;2,-d,q,s)$$

$$= -q \frac{i}{Z_o} I_f(2,-d,q,s;1,d,q,s)$$

$$= q \frac{i}{Z_o} \int_S [\tilde{\vec{F}}_{d_q}^{(1)}(s) \times \tilde{\vec{F}}_{-d_q}^{(2)}(s)] \cdot \vec{n} dS \tag{15.13}$$

This can be split into two parts by choosing $q = +1$, then $q = -1$, and adding and subtracting the results. In general the results are not as convenient as for both fields of the same type. Note, however, that the surface integral in equation 15.13 is independent of the choice of S as long as S contains V_o.

B. Energy: the case of $q_1 = -q_2 \equiv q$

Equation 13.11 becomes

$$I_\infty = \int_{S_\infty} [\tilde{\vec{F}}_q^{(1)}(s) \times \tilde{\vec{F}}_{-q}^{(2)}(s)] \cdot \vec{n} dS$$

$$= qi2\gamma \int_{V_\infty} \tilde{\vec{F}}_q^{(1)}(s) \cdot \tilde{\vec{F}}_{-q}^{(2)}(s) dV$$

$$+ qiZ_o [R'_{c_q}(2,s;1,s') + R'_{c_{-q}}(1,s;2,s')] \quad (15.14)$$

$$R'_{c_q}(2,s;1,s') = \langle \tilde{\vec{F}}_q^{(2)}(s); \tilde{\vec{K}}_{-q}^{(1)}(s') \rangle$$

$$R'_{c_{-q}}(1,s;2,s') = \langle \tilde{\vec{F}}_{-q}^{(1)}(s); \tilde{\vec{K}}_q^{(2)}(s') \rangle$$

where s' is kept separate for a moment.

For both solutions retarded $I_\infty = 0$ for $Re[s] \geq \delta > 0$. However, the integral over V_O is not necessarily zero because of the presence of the integral over V_∞. For both solutions advanced $I_\infty = 0$ for $Re[s] \leq \delta < 0$ with similar consequences. If $Re[s] = 0$ then we consider

$$I_\infty = \left[\lim_{r\to\infty} e^{-2d\gamma r}\right] I_f$$

$$I_f(1,d,q,s;2,d,-q,s)$$

$$= \int_{S'} [[r\tilde{\vec{F}}_{d_{f_q}}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{F}}_{d_{f_{-q}}}^{(2)}(\vec{r},s)]] \cdot \vec{e}_r dS'$$

(15.15)

$$= 2(s\mu_o)^2 \int_{S'} \Big\{ [\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_q^{(1)}(s') \rangle \times \langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_{-q}^{(2)}(s') \rangle] \cdot \vec{e}_r$$

$$- qdi \langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_q^{(1)}(s') \rangle_T \cdot \langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_{-q}^{(2)}(s') \rangle_T \Big\} dS'$$

$$= -I_f(2,d,-q,s;1,d,q,s)$$

Using this result one can consider the calculation of I as $r \to \infty$ for $Re[s] = 0$; note the factor $e^{-2d\gamma r}$ has magnitude 1 but varying phase as $r \to \infty$.

For one solution retarded and the other advanced we have

$$I_\infty = I_f(1,d,q,s;2,-d,-q,s)$$

$$= \int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_q}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_{-q}}(\vec{r},s)]] \cdot \vec{e}_r dS'$$

$$= 0$$

$$= -I_f(2,-d,-q,s;1,d,q,s) \tag{15.16}$$

which is a very convenient result and which shows an advantage of sometimes mixing retarded and advanced solutions.

Now letting s' = s consider the case that both solutions are retarded or both advanced. For $Re[ds] \geq \delta > 0$ we have from equation 15.14

$$R'_{c_q}(2,d,s;1,d,s) + R'_{c_{-q}}(1,d,s;2,d,s)$$

$$= -s\epsilon_o \int_{V_\infty} \tilde{\vec{F}}^{(1)}_{d_q}(s) \cdot \tilde{\vec{F}}^{(2)}_{-d_{-q}}(s)\, dV \tag{15.17}$$

For $Re[s] = 0$ we have I_f as given by equation 15.15 which can be used in equation 15.14 for large r. Note that r should in general still be specified because of the phase factor $e^{-2\gamma dr}$. So unless I_f can be shown to be zero, say in special cases, the limit as $r \to \infty$ cannot be strictly taken but an asymptotic form can be found as above.

For the case of one solution retarded and the other advanced we have for all s a combined subreaction theorem for mixed retarded and advanced solutions as

$$R'_{c_q}(2,-d,s;1,d,s) + R'_{c_{-q}}(1,d,s;2,-d,s)$$

$$= -2s\epsilon_o \int_{V_\infty} \tilde{\vec{F}}^{(1)}_{d_q}(s) \cdot \tilde{\vec{F}}^{(2)}_{-d_{-q}}(s)\, dV \tag{15.18}$$

This can be split to give an electric subreaction theorem

$$R'_e(2,-d,s;1,d,s) + R'_e(1,d,s;2,-d,s)$$

$$= -2s\varepsilon_o \int_{V_\infty} [\tilde{\vec{E}}_d^{(1)}(s) \cdot \tilde{\vec{E}}_{-d}^{(2)}(s) + Z_o^2 \tilde{\vec{H}}_d^{(1)}(s) \cdot \tilde{\vec{H}}_{-d}^{(2)}(s)]dV \quad (15.19)$$

and a magnetic subreaction theorem

$$R'_m(2,-d,s;1,d,s) - R'_m(1,d,s;2,-d,s)$$

$$= -2\gamma \int_{V_\infty} [-\tilde{\vec{E}}_d^{(1)}(s) \cdot \tilde{\vec{H}}_{-d}^{(2)}(s) + \tilde{\vec{H}}_d^{(1)}(s) \cdot \tilde{\vec{E}}_{-d}^{(2)}(s)]dV \quad (15.20)$$

Reviewing the results of this section note that for reciprocity theorems having both solutions retarded or both advanced gave the simplest results. On the other hand for the energy theorems having mixed retarded and advanced solutions gave the simplest results. Again note that for this section there is only one complex frequency, s.

XVI. Reduction to the Case That the Two Sets of Current Densities are the Same with Identical Frequencies for Infinite Boundary Surface

Now let the 1 and 2 current densities be identical and let all complex frequencies be the same, s. If the combined fields are both retarded or both advanced then the fields are also identical. If, however, one combined field is retarded and the other advanced then the fields are not in general the same except for special cases (such as cavity modes or identically zero current density).

A. Reciprocity: the case of $q_1 = q_2 \equiv q$

First consider the case that both combined fields are retarded or both are advanced. Then the reaction theorems (equations 15.7 through 15.9) and the surface integrals (equations 15.10 through 15.13) all reduce to tautologies. They need not be considered here further.

Second consider the case that one solution is retarded and the other advanced. Equation 15.13 reduces to

$$R_{c_q}(1,d,s;1,-d,s) - R_{c_q}(1,-d,s;1,d,s)$$

$$= \left\langle \tilde{\vec{F}}_{d_q}(s) - \tilde{\vec{F}}_{-d_q}(s); \tilde{\vec{K}}_q(s) \right\rangle$$

$$= q\,\frac{i}{Z_o}\,I_f(1,d,q,s;1,-d,q,s)$$

$$= -q\,\frac{i}{Z_o}\,I_f(2,-d,q,s;1,d,q,s)$$

$$= 2(s\mu_o)^2 \int_{S'} \Big\{ \left[\left\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_q(s) \right\rangle \times \left\langle \tilde{g}_{-d_f}(s), \tilde{\vec{K}}_q(s) \right\rangle \right] \cdot \vec{e}_r$$

$$-qdi \left\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}_q(s) \right\rangle_T \cdot \left\langle \tilde{g}_{-d_f}(s), \tilde{\vec{K}}_q(s) \right\rangle_T \Big\} dS'$$

$$= q\,\frac{i}{Z_o} \int_S \left[\tilde{\vec{F}}_{d_q}(s) \times \tilde{\vec{F}}_{-d_q}(s) \right] \cdot \vec{n}\, dS \qquad (16.1)$$

where the superscripts are now removed (being the same). Note that the above theorem is basically one for sum and difference fields (between retarded and advanced).

B. Energy: the case of $q_1 = -q_2 \equiv q$

Let the combined fields be both retarded or both advanced. For $\mathrm{Re}[ds] \geq \delta > 0$ equation 15.17 gives

$$R'_{c_q}(1,d,s;1,d,s) + R'_{c_{-q}}(1,d,s;1,d,s)$$

$$= 2\left[\left\langle \tilde{\vec{E}}_d(s);\tilde{\vec{J}}(s)\right\rangle + \left\langle \tilde{\vec{H}}_d(s);\tilde{\vec{J}}_m(s)\right\rangle\right]$$

$$= 2\, R'_e(1,d,s;1,d,s)$$

$$= -2s\varepsilon_o \int_{V_\infty} \tilde{\vec{F}}_{d_q}(s) \cdot \tilde{\vec{F}}_{d_{-q}}(s)\,dV$$

$$= -2s\varepsilon_o \int_{V_\infty} [\tilde{\vec{E}}_d(s) \cdot \tilde{\vec{E}}_d(s) + \tilde{\vec{H}}_d(s) \cdot \tilde{\vec{H}}_d(s)]\,dV \qquad (16.2)$$

This is the energy theorem for a single solution of Maxwell's equations for the case $\mathrm{Re}[ds] \geq \delta > 0$. For $\mathrm{Re}[s] = 0$ the integral over the current densities and fields simplifies by cancellation of some terms, but the exponential term $e^{-2d\gamma r}$ still needs to be included with the results.

Let the combined fields be one retarded and the other advanced. Then equation 15.18 gives, for all s,

$$R'_{c_q}(1,-d,s;1,d,s) + R'_{c_{-q}}(1,d,s;2,-d,s)$$

$$= \left\langle \tilde{\vec{F}}_{-d_q}(s);\tilde{\vec{K}}_{-q}(s)\right\rangle + \left\langle \tilde{\vec{F}}_{d_{-q}}(s);\tilde{\vec{K}}_q(s)\right\rangle$$

$$= 2s\varepsilon_o \int_{V_\infty} \tilde{\vec{F}}_{d_q}(s) \cdot \tilde{\vec{F}}_{-d_{-q}}(s)\,dV \qquad (16.3)$$

This is the combined energy theorem for a single current density with mixed retarded and advanced solutions. This can be split to give an energy theorem

$$R'_e(1,-d,s;1,d,s) + R'_e(1,d,s;1,-d,s)$$

$$= \left\langle \vec{\tilde{E}}_d(s) + \vec{\tilde{E}}_{-d}(s) ; \vec{\tilde{J}}(s) \right\rangle + \left\langle \vec{\tilde{H}}_d(s) + \vec{\tilde{H}}_{-d}(s) ; \vec{\tilde{J}}_m(s) \right\rangle$$

$$= 2s\varepsilon_o \int_{V_\infty} [\vec{\tilde{E}}_d(s) \cdot \vec{\tilde{E}}_{-d}(s) + Z_o^2 \vec{\tilde{H}}_d(s) \cdot \vec{\tilde{H}}_{-d}(s)] dV \qquad (16.4)$$

and an interchanged energy theorem

$$R'_m(1,-d,s;1,d,s) - R'_m(1,d,s;1,-d,s)$$

$$= \frac{1}{Z_o} \left\langle \vec{\tilde{E}}_d(s) - \vec{\tilde{E}}_{-d}(s) ; \vec{\tilde{J}}_m(s) \right\rangle - Z_o \left\langle \vec{\tilde{H}}_d(s) - \vec{\tilde{H}}_{-d}(s) ; \vec{\tilde{J}}(s) \right\rangle$$

$$= 2\gamma \int_{V_\infty} [-\vec{\tilde{E}}_d(s) \cdot \vec{\tilde{H}}_{-d}(s) + \vec{\tilde{H}}_d(s) \cdot \vec{\tilde{E}}_{-d}(s)] dV \qquad (16.5)$$

Thus the case of mixed retarded and advanced fields for a single current density gives some convenient theorems for sum and difference fields with respect to retarded and advanced.

XVII. Combined Reciprocity and Energy with Opposite Frequencies for Infinite Boundary Surface

Initially choose

$$s_1 = -s_2 \equiv s \ , \qquad s_1' = -s_2' \equiv s' \tag{17.1}$$

Then the surface integral I_f from section XIII can again be used to obtain for reaction and subreaction quantities and for the surface integral I for general S (containing V_o).

A. Energy for opposite frequencies: the case of $q_1 = q_2 \equiv q$

Equation 13.11 gives

$$\begin{aligned} I_\infty &= \int_{S_\infty} [\tilde{\vec{F}}_q(s) \times \tilde{\vec{F}}_q(-s)] \cdot \vec{n} dS \\ &= qi2\gamma \int_{V_\infty} \tilde{\vec{F}}_q(s) \cdot \tilde{\vec{F}}_q(-s) dV \\ &\quad + qiZ_o [R'_{c_q}(2,-s;1,s') - R'_{c_q}(1,s;2,-s')] \end{aligned} \tag{17.2}$$

$$R'_{c_q}(2,-s;1,s) = \left\langle \tilde{\vec{F}}_q^{(2)}(-s) ; \tilde{\vec{K}}_q^{(1)}(s') \right\rangle$$

$$R'_{c_q}(1,s;2,-s) = \left\langle \tilde{\vec{F}}_q^{(1)}(s) ; \tilde{\vec{K}}_q^{(2)}(-s') \right\rangle$$

For both solutions retarded or both advanced we have

$$\begin{aligned} I_\infty &= I_f(1,d,q,s;2,d,q,-s) \\ &= \int_{S'} \left[[r\tilde{\vec{F}}_{d_q}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{F}}_{d_q}^{(2)}(\vec{r},-s)] \right] \cdot \vec{e}_r dS' \\ &= 0 \\ &= -I_f(2,d,q,-s;1,d,q,s) \end{aligned} \tag{17.3}$$

For one solution retarded and the other advanced we have

$$I_\infty = \left[\lim_{r\to\infty} e^{-2d\gamma r}\right] I_f$$

$$\begin{aligned}
I_f(1,d,q,s;2,-d,q,-s) &= \int_{S'} \left[[r\tilde{\vec{F}}^{(1)}_{d_q}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_q}(\vec{r},-s)]\right] \cdot \vec{e}_r dS' \\
&= -2(s\mu_o)^2 \int_{S'} \Big\{ [\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}^{(1)}_q(s')\rangle \\
&\quad \times \langle \tilde{g}_{-d_f}(-s), \tilde{\vec{K}}^{(2)}_q(-s')\rangle] \cdot \vec{e}_r \\
&\quad - qd_i \langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}^{(1)}_q(s')\rangle_T \cdot \langle \tilde{g}_{-d_f}(-s), \tilde{\vec{K}}^{(2)}_q(-s')\rangle_T \Big\} dS'
\end{aligned} \tag{17.4}$$

For $\mathrm{Re}[ds] \geq \delta > 0$ then $I_\infty = 0$. This result can also be applied to the case of $\mathrm{Re}[s] = 0$. However, the factor $e^{-2d\gamma r}$ with magnitude 1 and varying phase as $r \to \infty$ needs to be included.

Now letting $s = s'$ consider the case that both solutions are retarded or both advanced. For all s we have a combined subreaction theorem as

$$R'_{c_q}(2,d,-s;1,d,s) - R'_{c_q}(1,d,s;2,d,-s)$$

$$= -2s\varepsilon_o \int_{V_\infty} \tilde{\vec{F}}^{(1)}_{d_q}(s) \cdot \tilde{\vec{F}}^{(2)}_{d_q}(-s)\, dV \tag{17.5}$$

This can be split into an electric subreaction theorem

$$R'_e(2,d,-s;1,d,s) - R'_e(1,d,s;2,d,-s)$$

$$= -2s\varepsilon_o \int_{V_\infty} [\tilde{\vec{E}}^{(1)}_d(s) \cdot \tilde{\vec{E}}^{(2)}_d(-s) - Z_o^2 \tilde{\vec{H}}^{(1)}_d(s) \cdot \tilde{\vec{H}}^{(2)}_d(-s)]\, dV \tag{17.6}$$

and a magnetic subreaction theorem

$$R'_m(2,d,-s;1,d,s) - R'_m(1,d,s;2,d,-s)$$

$$= -2\gamma \int_{V_\infty} [\vec{\tilde{E}}_d^{(1)}(s) \cdot \vec{\tilde{H}}_d^{(2)}(-s) + \vec{\tilde{H}}_d^{(1)}(s) \cdot \vec{\tilde{E}}_d^{(2)}(-s)]dV \quad (17.7)$$

For the case of one solution retarded and the other advanced with $Re[ds] \geq \delta > 0$ we have

$$R'_{c_q}(2,-d,-s;1,d,s) - R'_{c_q}(1,d,s;2,-d,-s)$$

$$= -2s\varepsilon_o \int_{V_\infty} \vec{\tilde{F}}_{d_q}^{(1)}(s) \cdot \vec{\tilde{F}}_{-d_q}^{(2)}(-s)dV \quad (17.8)$$

For $Re[s] = 0$ we have I_f as given by equation 17.4; this can be used in equation 17.2 but r must in general still be specified.

B. Reciprocity for opposite frequencies: the case of $q_1 = -q_2 \equiv q$

Equation 13.8 gives

$$I = I_\infty = \int_{S_\infty} [\vec{\tilde{F}}_q^{(1)}(s) \times \vec{\tilde{F}}_{-q}^{(2)}(-s)] \cdot \vec{n}dS$$

$$= qiZ_o[\langle \vec{\tilde{F}}_{-q}^{(2)}(-s);\vec{\tilde{K}}_q^{(1)}(s)\rangle + \langle \vec{\tilde{F}}_q^{(1)}(s);\vec{\tilde{K}}_{-q}^{(2)}(-s)\rangle]$$

$$= qiZ_o[R_{c_{-q}}(2,-s;1,s) + R_{c_q}(1,s;2,-s)] \quad (17.9)$$

For both solutions retarded or both advanced we have

$$I = I_\infty = I_f(1,d,q,s;2,d,-q,-s)$$

$$= \int_{S'} [[r\vec{\tilde{F}}_{d_q}^{(1)}(\vec{r},s)] \times [r\vec{\tilde{F}}_{d_{-q}}^{(2)}(\vec{r},-s)]] \cdot \vec{e}_r dS'$$

$$= -2(s\mu_o)^2 \int_{S'} \{[\langle \tilde{g}_{d_f}(s),\vec{\tilde{K}}_q^{(1)}(s')\rangle \times \langle \tilde{g}_{d_f}(-s),\vec{\tilde{K}}_{-q}(-s')\rangle] \cdot \vec{e}_r$$

$$-qdi\langle \tilde{g}_{d_f}(s), \tilde{\vec{K}}^{(1)}(s')\rangle_T \cdot \langle \tilde{g}_{d_f}(-s), \tilde{\vec{K}}_{-q}(-s')\rangle_T \Big\} dS'$$

$$= -I_f(2,d,-q,-s;1,d,q,s) \tag{17.10}$$

For one solution retarded and the other advanced we have

$$I = I_\infty = \left[\lim_{r\to\infty} e^{-2d\gamma r}\right] I_f$$

$$I_f(1,d,q,s;2,-d,-q,-s)$$

$$= \int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_q}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_{-q}}(\vec{r},s)]] \cdot \vec{e}_r dS' \tag{17.11}$$

$$= 0$$

$$= -I_f(2,-d,-q,-s;1,d,q,s)$$

For $Re[ds] \geq 0$ then $I_\infty = 0$. By analytic continuation on the integral over V_o then $I = 0$ for all s as long as S contains V_o.

Letting $s = s'$ consider the case that both solutions are retarded or both advanced. For all s we have a combined reaction theorem as

$$R_{c_{-q}}(2,d,-s;1,d,s) + R_{c_q}(1,d,s;2,d,-s)$$

$$= -q\frac{i}{Z_o} I_f(1,d,q,s;2,d,-q,-s)$$

$$= q\frac{i}{Z_o} I_f(2,d,-q,-s;1,d,q,s)$$

$$= -q\frac{i}{Z_o}\int_S [\tilde{\vec{F}}^{(1)}_{d_q}(s) \times \tilde{\vec{F}}^{(2)}_{d_{-q}}(-s)] \cdot \vec{n} dS \tag{17.12}$$

The integral over S is independent of the choice of S as long as it contains V_o. This can be split into two parts but is not as convenient as the mixed advanced and retarded case.

For the case of one solution retarded and the other advanced for all s we have a combined reaction theorem

$$R_{c_{-q}}(2,-d,-s;1,d,s) + R_{c_q}(1,d,s;2,-d,-s) = 0 \tag{17.13}$$

This can be split into an electric reaction theorem

$$R_e(2,-d,-s;1,d,s) + R_e(1,d,s;2,-d,-s) = 0 \tag{17.14}$$

and a magnetic reaction theorem

$$-R_m(2,-d,-s;1,d,s) + R_m(1,d,s;2,-d,-s) = 0 \tag{17.15}$$

For this case the surface integral (S enclosing V_o) is zero giving in combined form

$$\begin{aligned} &\int_S [\tilde{\vec{F}}^{(1)}_{d_q}(s) \times \tilde{\vec{F}}^{(2)}_{-d_{-q}}(-s)] \cdot \vec{n}dS = 0 \\ &\int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_{f_q}}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_{f_{-q}}}(\vec{r},-s)]] \cdot \vec{e}_r dS' = 0 \end{aligned} \tag{17.16}$$

which can be split to give

$$\int_S [\tilde{\vec{E}}^{(1)}_d(s) \times \tilde{\vec{E}}^{(2)}_{-d}(-s) + Z_o^2\tilde{\vec{H}}^{(1)}_d(s) \times \tilde{\vec{H}}^{(2)}_{-d}(-s)] \cdot \vec{n}dS = 0$$

$$\begin{aligned} &\int_{S'} [[r\tilde{\vec{E}}^{(1)}_{df}(\vec{r},s)] \times [r\tilde{\vec{E}}^{(2)}_{-d_f}(\vec{r},-s)] \\ &\quad + Z_o^2[r\tilde{\vec{H}}^{(1)}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{H}}^{(2)}_{-d_f}(\vec{r},-s)]] \cdot \vec{e}_r dS = 0 \end{aligned} \tag{17.17}$$

and

$$\int_S [\tilde{\vec{E}}_d^{(1)}(s) \times \tilde{\vec{H}}_{-d}^{(2)}(-s) - \tilde{\vec{H}}_d^{(1)}(s) \times \tilde{\vec{E}}_{-d}^{(2)}(-s)] \cdot \vec{n} dS = 0$$

$$\int_{S'} \Big[[r\tilde{\vec{E}}_{d_f}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{H}}_{-d_f}^{(2)}(\vec{r},-s)] \qquad (17.18)$$

$$- [r\tilde{\vec{H}}_{d_f}^{(1)}(\vec{r},s)] \times [r\tilde{\vec{E}}_{-d_f}^{(2)}(\vec{r},-s)]\Big] \cdot \vec{e}_r dS = 0$$

XVIII. Reduction to the Case That the Two Sets of Current Densities are the Same Except for Opposite Frequencies for Infinite Boundary Surface

Now let the 1 and 2 current densities be the same except for being evaluated for opposite frequencies. The superscripts may then be suppressed.

A. Energy for opposite frequencies: the case of $q_1 = q_2 \equiv q$

First consider the case that both solutions are retarded or both advanced. Then for all s we have from equation 17.5 the result

$$R'_{c_q}(1,d,-s;1,d,s) - R'_{c_q}(1,d,s;1,d,-s)$$

$$= \langle \tilde{\vec{F}}_{d_q}(-s);\tilde{\vec{K}}_q(s)\rangle - \langle \tilde{\vec{F}}_{d_q}(s);\tilde{\vec{K}}_q(-s)\rangle$$

$$= -2s\varepsilon_o \int_{V_\infty} \tilde{\vec{F}}_{d_q}(s) \cdot \tilde{\vec{F}}_{d_q}(-s)\,dV \qquad (18.1)$$

which is a combined energy theorem for opposite frequencies. This can be split to give an opposite frequencies energy theorem

$$R'_e(1,d,-s;1,d,s) - R_e(1,d,s;1,d,-s)$$

$$= [-\langle \tilde{\vec{E}}_d(s);\tilde{\vec{J}}(-s)\rangle + \langle \tilde{\vec{H}}_d(s);\tilde{\vec{J}}_m(-s)\rangle]$$

$$+ [\langle \tilde{\vec{E}}_d(-s);\tilde{\vec{J}}(s)\rangle - \langle \tilde{\vec{H}}_d(-s);\tilde{\vec{J}}_m(s)\rangle]$$

$$= -2s\varepsilon_o \int_{V_\infty} [\tilde{\vec{E}}_d(s) \cdot \tilde{\vec{E}}_d(-s) - Z_o^2 \tilde{\vec{H}}_d(s) \cdot \tilde{\vec{H}}_d(-s)]\,dV \qquad (18.2)$$

and an opposite frequencies interchanged energy theorem

$$R'_m(1,d,-s;1,d,s) - R'_m(1,d,s;1,d,-s)$$

$$= \left[-\frac{1}{Z_o} \left\langle \tilde{\vec{E}}_d(s);\tilde{\vec{J}}_m(-s) \right\rangle - Z_o \left\langle \tilde{\vec{H}}_d(s);\tilde{\vec{J}}(-s) \right\rangle \right]$$

$$+ \left[\frac{1}{Z_o} \left\langle \tilde{\vec{E}}_d(-s);\tilde{\vec{J}}_m(s) \right\rangle + Z_o \left\langle \tilde{\vec{H}}_d(-s);\tilde{\vec{J}}(s) \right\rangle \right] .$$

$$= -2\gamma \int_{V_\infty} [\tilde{\vec{E}}_d(s) \cdot \tilde{\vec{H}}_d(-s) + \tilde{\vec{H}}_d(s) \cdot \tilde{\vec{H}}_d(-s)]dV \qquad (18.3)$$

Let the combined fields be one retarded and one advanced. For $Re[ds] \geq \delta > 0$ we have

$$R'_{c_q}(1,-d,-s;1,d,s) - R'_{c_q}(1,d,s;1,-d,-s)$$

$$= \left\langle \tilde{\vec{F}}_{-d_q}(-s);\tilde{\vec{K}}_q(s) \right\rangle - \left\langle \tilde{\vec{F}}_{d_q}(s);\tilde{\vec{K}}_q(-s) \right\rangle$$

$$= -2s\epsilon_o \int_{V_\infty} \tilde{\vec{F}}_{d_q}(s) \cdot \tilde{\vec{F}}_{-d_q}(-s)dV \qquad (18.4)$$

For $Re[s] = 0$ an additional term I_f from equation 17.4 is needed but r must in general be specified for the outer boundary.

B. Reciprocity for opposite frequencies: the case of $q_1 = -q_2 \equiv q$

Let the sets of fields be both retarded or both advanced. Then for all s equation 17.12 reduces to a combined reciprocity theorem for opposite frequencies as

$$R_{c_{-q}}(1,d,-s;1,d,s) + R_{c_q}(1,d,s;1,d,-s)$$

$$= \left\langle \tilde{\vec{F}}_{d_{-q}}(-s);\tilde{\vec{K}}_q(s) \right\rangle + \left\langle \tilde{\vec{F}}_{d_q}(s);\tilde{\vec{K}}_q(-s) \right\rangle$$

$$= -q\frac{i}{Z_o} \int_S [\tilde{\vec{F}}_{d_q}(s) \times \tilde{\vec{F}}_{d_{-q}}(-s)] \cdot \vec{n}dS \qquad (18.5)$$

For the case of one set of fields retarded and the other advanced we have the combined reaction theorem for opposite frequencies

$$R_{c_{-q}}(1,-d,-s;1,d,s) + R_{c_q}(1,d,s;1,-d,-s) = 0$$

$$\left\langle \tilde{\vec{F}}_{-d_{-q}}(-s);\tilde{\vec{K}}_q(s)\right\rangle + \left\langle \tilde{\vec{F}}_{d_q}(s);\tilde{\vec{K}}_{-q}(-s)\right\rangle = 0 \tag{18.6}$$

which is valid for all s. The associated opposite frequencies electric reciprocity theorem is

$$R_e(1,-d,-s;1,d,s) + R_e(1,d,s;1,-d,-s) = 0$$

$$\left\langle \tilde{\vec{E}}_d(s);\tilde{\vec{J}}(-s)\right\rangle + \left\langle \tilde{\vec{H}}_d(s);\tilde{\vec{J}}_m(-s)\right\rangle \tag{18.7}$$

$$+ \left\langle \tilde{\vec{E}}_{-d}(-s);\tilde{\vec{J}}(s)\right\rangle + \left\langle \tilde{\vec{H}}_{-d}(-s);\tilde{\vec{J}}_m(s)\right\rangle = 0$$

and the opposite frequencies magnetic reciprocity theorem is

$$-R_m(1,-d,-s;1,d,s) + R_m(1,d,s;1,-d,-s) = 0$$

$$\left\langle \tilde{\vec{E}}_d(s);\tilde{\vec{J}}_m(-s)\right\rangle - Z_o^2\left\langle \tilde{\vec{H}}_d(s);\tilde{\vec{J}}(-s)\right\rangle \tag{18.8}$$

$$- \left\langle \tilde{\vec{E}}_{-d}(-s);\tilde{\vec{J}}_m(s)\right\rangle + Z_o^2\left\langle \tilde{\vec{H}}_{-d}(-s);\tilde{\vec{J}}(s)\right\rangle = 0$$

Note that the surface integrals in equations 17.16 through 17.18 can be directly interpreted for cases that the 1 and 2 fields are made the "same" (i.e. have the same sources in this case). These are

$$\int_S [\tilde{\vec{F}}_{d_q}(s) \times \tilde{\vec{F}}_{-d_{-q}}(-s)] \cdot \vec{n}\, dS = 0$$

$$\int_{S'} [[r\tilde{\vec{F}}^{(1)}_{d_{f_q}}(\vec{r},s)] \times [r\tilde{\vec{F}}^{(2)}_{-d_{f_{-q}}}(\vec{r},-s)]] \cdot \vec{e}_r\, dS' = 0 \tag{18.9}$$

which can be split to give

$$\int_S [\tilde{\vec{E}}_d(s) \times \tilde{\vec{E}}_{-d}(-s) + Z_o^2 \tilde{\vec{H}}_d(s) \times \tilde{\vec{H}}_{-d}(-s)] \cdot \vec{n} dS = 0$$

$$\int_{S'} \Big[[r\tilde{\vec{E}}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{E}}_{-d_f}(\vec{r},-s)] + Z_o^2 [r\tilde{\vec{H}}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{H}}_{-d_f}(\vec{r},-s)] \Big] \cdot \vec{e}_r dS = 0 \tag{18.10}$$

and

$$\int_S [\tilde{\vec{E}}_d(s) \times \tilde{\vec{H}}_{-d}(-s) - \tilde{\vec{H}}_d(s) \times \tilde{\vec{E}}_{-d}(-s)] \cdot \vec{n} dS = 0$$

$$\int_{S'} \Big[[r\tilde{\vec{E}}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{H}}_{-d_f}(\vec{r},-s)] - [r\tilde{\vec{H}}_{d_f}(\vec{r},s)] \times [r\tilde{\vec{E}}_{-d_f}(\vec{r},-s)] \Big] \cdot \vec{e}_r dS = 0 \tag{18.11}$$

C. Combinations of previous results

Note that in the results of parts A and B of this section there are some common terms. This allows substitutions to be made so that other forms of equations are obtained. In doing this more complicated combinations of d and q subscripts are produced and sum and difference fields (with respect to advanced and retarded) can be exhibited.

XIX. Summary of the Form of the Results

A lot of territory has been covered in the various cases of the reciprocity/energy theorems in this note. At this point let us consider where we have arrived.

There are some patterns in the forms of the results. The forms of the various theorems are invariant to certain transformations. For example, the 1 and 2 fields can be interchanged with only a flip of sign due to the cross product of the two field solutions which appears in the equations. Which field is 1 and which is 2 is clearly arbitrary.

In the various equations in this note there are considered the various combinations of frequencies, combined field index q, and retarded/advanced solutions for the two fields and associated current densities. Let us assign values of ± 1 to each of three variables which we write as a vector (c_s, c_q, c_d). Define these by

$$\vec{c} \equiv (c_s, c_q, c_d)$$

$$c_s \equiv \begin{cases} +1 & \text{for } s_1 = s_2 \equiv s \\ -1 & \text{for } s_1 = -s_2 \equiv s \end{cases}$$

$$c_q \equiv \begin{cases} +1 & \text{for } q_1 = q_2 \equiv q \\ -1 & \text{for } q_1 = -q_2 \equiv q \end{cases} \tag{19.1}$$

$$c_d \equiv \begin{cases} +1 & \text{for } d_1 = d_2 \equiv d \\ -1 & \text{for } d_1 = -d_2 \equiv d \end{cases}$$

These can also be written as

$$c_s = \frac{s_2}{s_1} = \frac{s_2}{s}$$

$$c_q = \frac{q_2}{q_1} = \frac{q_2}{q} \tag{19.2}$$

$$c_d = \frac{d_2}{d_1} = \frac{d_2}{d}$$

This gives eight possible choices for $\vec{c}$, each of which is associated with particular equations in this note.

Define a set of eight transformation matrices as

$$\overleftrightarrow{\tau} \equiv \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{19.3}$$

depending on the choices of the signs of each of the diagonal elements. Multiplying a particular $\vec{c}$ by a particular $\overleftrightarrow{\tau}$ (other than the identity) gives a different $\vec{c}$.

The eight possible choices for $\overleftrightarrow{\tau}$ form a group with eight elements under the operation of matrix multiplication. One choice of $\overleftrightarrow{\tau}$ is the identity as

$$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19.4}$$

and each element is its own inverse so that the matrices are involutory as

$$\overleftrightarrow{\tau} \cdot \overleftrightarrow{\tau}^{-1} = \overleftrightarrow{I} \, , \qquad \overleftrightarrow{\tau}^{-1} = \overleftrightarrow{\tau} \tag{19.5}$$

for each particular choice of $\overleftrightarrow{\tau}$. Note that these matrices are unitary with determinant equal to ± 1. Furthermore this group is Abelian since the product of diagonal matrices is commutative.

This group can be referred to as the reciprocity and energy group. Consider first the case where the volume of concern has finite dimensions as treated in sections III through XI. For simplicity then consider $\vec{c}$ with 2 components and $\overleftrightarrow{\tau}$ as a 2 × 2 matrix with only s and q indices used since the retarded vs. advanced split does not apply to this case. The choices of $\vec{c}$ are

$$\begin{matrix} (1, 1) & (1, -1) \\ (-1, 1) & (-1, -1) \end{matrix} \tag{19.6}$$

The associated transformation matrices $\overleftrightarrow{\tau}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (19.7)$$

and these form a group which may be referred to as the reciprocity and energy group for volumes of finite dimensions.

From the cases considered for volumes with finite dimensions we have choices of $\vec{c}$ for reciprocity (equations 4.1 and 9.4) as

$$(1,\ 1) \qquad (-1,\ -1) \qquad (19.8)$$

characterized by

$$c_s c_q = +1 \qquad (19.9)$$

and choices of $\vec{c}$ for energy (equations 4.4 and 9.1) as

$$(1,\ -1) \qquad (-1,\ 1) \qquad (19.10)$$

characterized by

$$c_s c_q = -1 \qquad (19.11)$$

For each of these sets of choices of $\vec{c}$ there is a set of transformation matrices $\overset{\leftrightarrow}{t}$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (19.12)$$

which also form a group (an S_2 group[3]). This group is an Abelian subgroup of the reciprocity and energy group. It can be referred to as the reciprocity group and as the energy group (for volumes of finite dimensions). Applying the transformations of equations 19.12 to the vectors of equations 19.8 and 19.10 produces vectors of the same type (reciprocity or energy) as those begun with. The remaining transformation matrices $\overset{\leftrightarrow}{t}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{19.13}$$

which transform reciprocity $\vec{c}$ to energy $\vec{c}$ and conversely. They do not form a group, however, as they do not include the identity.

For 3 indices (or components) for $\vec{c}$ and 3 × 3 form of the reciprocity and energy group has eight elements as in equation 19.3 associated with eight values of $\vec{c}$ as in equation 19.1. For reciprocity the choices of $\vec{c}$ are

$$\begin{matrix} (1, 1, 1) & (1, 1, -1) \\ (-1, -1, 1) & (-1, -1, -1) \end{matrix} \tag{19.14}$$

and for energy the choices of $\vec{c}$ are

$$\begin{matrix} (1, -1, 1) & (1, -1, -1) \\ (-1, 1, 1) & (-1, 1, -1) \end{matrix} \tag{19.15}$$

For each of these sets of choices of $\vec{c}$ there is a set of transformation matrices $\overleftrightarrow{t}$ as

$$\begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{matrix} \tag{19.16}$$

which form a group which is an Abelian subgroup of the reciprocity and energy group (equation 19.3). Multiplying the $\vec{c}$ values in equations 19.14 and 19.15 by these $\overleftrightarrow{t}$ matrices gives $\vec{c}$ values in the same respective sets. This group (equation 19.16) can be referred to as both the reciprocity group and the energy group. The remaining transformation matrices $\overleftrightarrow{t}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{19.17}$$

These do not form a group. They transform $\vec{c}$ values in equation 19.14 to those in equation 19.15 and conversely.

There are other patterns which can be seen in the results. The following table summarizes some of the results for the various $\vec{c}$ values as found in sections XV and XVII.

$\vec{c} = (c_s, c_q, c_d)$	I_f	I_∞, I	reciprocity or energy
(1, 1, 1)	$I_f = 0$	$I_\infty = I = 0$ (all s by analytic continuation)	reciprocity
(1, 1, -1)	$I_f \neq 0$	$I_\infty = I = I_f \neq 0$ (all s)	reciprocity
(1, -1, 1)	$I_f \neq 0$	$I_\infty = 0$ ($\mathrm{Re}[ds] \geq \delta > 0$)	energy
(1, -1, -1)	$I_f = 0$	$I_\infty = 0$ (all s)	energy
(-1, 1, 1)	$I_f = 0$	$I_\infty = 0$ (all s)	energy
(-1, 1, -1)	$I_f \neq 0$	$I_\infty = 0$ ($\mathrm{Re}[ds] \geq \delta > 0$)	energy
(-1, -1, 1)	$I_f \neq 0$	$I_\infty = I = I_f \neq 0$ (all s)	reciprocity
(-1, -1, -1)	$I_f = 0$	$I_\infty = I = 0$ (all s by analytic continuation)	reciprocity

Table 19.1 Results summary for various choices of relative signs of frequency, q values, and retarded/advanced combinations for infinite volumes

This table shows various patterns among the results for the set of all choices for $\vec{c}$. One such pattern has already been discussed, specifically the split into reciprocity and energy cases together with the group of transformations within these cases, as well as the transformations between the two cases. Note that the split between reciprocity depends on whether or not the integral of the dot product of two combined fields over V_∞ is zero.

Another pattern revealed in table 19.1 regards the far-field integral I_f. Half of the cases have $I_f = 0$ for which the $\vec{c}$ values are

$$\begin{matrix} (1, 1, 1) & (1, -1, -1) \\ (-1, 1, 1) & (-1, -1, -1) \end{matrix} \tag{19.18}$$

characterized by

$$c_q c_d = 1 \tag{19.19}$$

For $I_f \neq 0$ the $\vec{c}$ values are

$$\begin{matrix} (1, 1, -1) & (1, -1, 1) \\ (-1, 1, -1) & (-1, -1, 1) \end{matrix} \tag{19.20}$$

characterized by

$$c_q c_d = -1 \tag{19.21}$$

For each of these sets of choices of $\vec{c}$ there is a set of transformation matrices $\overleftrightarrow{t}$ as

$$\begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{matrix} \tag{19.22}$$

which form a group which is an Abelian subgroup of the reciprocity and energy group. Multiplying the $\vec{c}$ values in equations 19.18 and 19.20 by these $\overleftrightarrow{t}$ matrices gives $\vec{c}$ values in the same respective sets. Note that the group elements in equation 19.22 are different from the group elements in equation 19.16, although each group has the same number of elements. The group in equation 19.22 can be referred to as the $I_f = 0$ group and the $I_f \neq 0$ group, or as the far field group for short. The remaining transformation matrices $\overleftrightarrow{t}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19.23}$$

which do not form a group but change $\vec{c}$ values in equation 19.18 to those in equation 19.20 and conversely.

Another pattern from in table 19.1 results in four sets of $\vec{c}$ values. Observe the pattern of the results for I_∞. This can be summarized in four sets of $\vec{c}$ values as

$I_\infty = I = 0$

(all s by analytic continuation)

$\vec{c} = (1, 1, 1)$, $(-1, -1, -1)$

$I_\infty = I = I_f \neq 0$

(all s)

$\vec{c} = (1, 1, -1)$, $(-1, -1, 1)$

$I_\infty = 0$

($\mathrm{Re}[ds] \geq \delta > 0$)

$\vec{c} = (1, -1, 1)$, $(-1, 1, -1)$

$$I_\infty = 0 \qquad (19.24)$$

(all s)

$$\vec{c} = (1, -1, -1) \;, \qquad (-1, 1, 1)$$

For each of these 4 sets of choices of $\vec{c}$ there is a set of transformation matrices $\vec{\vec{t}}$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (19.25)$$

which form an S_2 subgroup which is an Abelian subgroup of the reciprocity and energy group. Multiplying a $\vec{c}$ value in equation 19.24 by either of these matrices results in a $\vec{c}$ value from the same set as the original $\vec{c}$ value. The remaining transformation matrices $\vec{\vec{t}}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(19.26)$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which do not form a group but change $\vec{c}$ values in equation 19.24 from one set to another.

Thus the reciprocity and energy group (equation 19.3) with eight elements has three subgroups (equations 19.16, 19.22, and 19.25) which give transformations which keep $\vec{c}$ values in their same sets. There are several ways of defining such sets of $\vec{c}$ values from table 19.1.

XX. Epilogue

Horatio
Oh, day and night, but this is wondrous strange!

Hamlet
And therefore as a stranger give it welcome.
There are more things in Heaven and earth, Horatio,
Than are dreamt of in your philosophy

William Shakespeare,
The Tragedy of Hamlet, Prince of Denmark
Act I, Scene V

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