Circuit and Electromagnetic System Design Notes

Note 42

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Calculation of the Inductance of a Circular Coil Consisting of any Number of Coaxial Turns, Each Individual Turn being of any Wire Size Located in Parallel Planes and of any Spacing to the Other Turns

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Abstract

This is the first in a series of notes on the calculation of inductance. This note deals with the calculation of the inductance of a circular coil consisting of any number of coaxial turns; each individual turn may consist of any wire size and of any spacing to the adjacent turns. Future notes will deal with the calculation of the inductance of various geometric configurations as found in Grover's classic book[1]. The advantage offered by the method in these notes is the elimination of the tedious interpolations from tables as used in Grover. Future notes will also detail the calculation of air core transformer self and mutual inductance for several configurations. The calculation of current distribution and the effect on inductance will also be included.

The calculation of the inductance of a circular coil is accomplished using MATHCAD PLUS 6.0. In-order to use the calculation method herein one must have a registered version of MATHCAD PLUS 6.0 or higher from MathSoft Inc. Cambridge, Massachusetts. Other computational mathematics programs can also be adapted to the formulation. The basic physics of the inductance calculations relies heavily on Smythe[2]. The coil consists of coaxial turns of round cross-section wire. Each turn may have a different wire size, turn diameter and spacing from adjacent turns. The procedure determines the complete symmetric inductance matrix, which consists of the self-inductance's of each turn as the main diagonal and all of the mutual inductance's between the turns as the symmetric elements. The total inductance of the coil, that is the inductance of all of the turns connected in series, is the sum of all of the matrix elements. A special case is the single layer solenoid.

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1 Grover, Frederick W., "Inductance Calculations, Working Formulas and Tables", Dover Publications, 1962.

2 Smythe, William R., "Static and Dynamic Electricity", McGraw-Hill , 1968.

INTRODUCTION

The basic building block of a circular cylindrical coil is the single turn or loop of wire. To evaluate the total inductance, the self-inductance of a loop is first determined as a function of the wire diameter and the loop diameter. The coil consists of a number, N, of coaxial loops connected in series. The mutual inductance's between all of the loops contribute to the total coil inductance. A general expression for the mutual inductance between two coaxial loops as a function of the mean diameters of the loops and the axial spacing is determined and this is used to calculate the entire array of mutual inductance's between all of the coil's turns. In the general case the cross-section of the coil may be of any shape, rectangular , etc. The simplest special case is the single layer solenoid. As long as all of the turns are coaxial, the wire size and diameter (radius) of each turn is specified, as well as the axial position of each turn; this method can be used to calculate the total inductance. MKS units are used throughout; $\mu_0 = 4\pi 10^{-7}$.

1. Basic Theory. In general, the vector magnetic potential, **A,** at a point r from a current element I**ds** is given by:

(1)
$$\mathbf{A} = \frac{\mu}{4\pi}\oint \frac{I\mathbf{ds}}{r}$$

The mutual inductance between two circuits, 1 and 2, is defined as the flux, Φ_{12}, through circuit 1 due to a unit current in circuit 2, or visa versa. The flux or **B.n** field integrated over the area of any closed circuit is equal to the line integral of the magnetic vector potential around that circuit. The mutual inductance, M_{12}, is therefore given by[3]:

(2)
$$M_{12} = M_{21} = \int_{S_1} \mathbf{B}_2 \bullet \mathbf{n}\, dS_1 = \oint \mathbf{A}_2 \bullet \mathbf{ds}_1$$

In principle equation (2) can be applied to any two circuits to obtain the mutual inductance or to a single circuit to obtain the self inductance.

1.1 Mutual inductance between two coaxial loops. The special case of two coaxial loops is illustrated in Fig.1[4].

[3] Smythe, op cit, p.333

[4] Smythe, op cit, p.290 and p.335

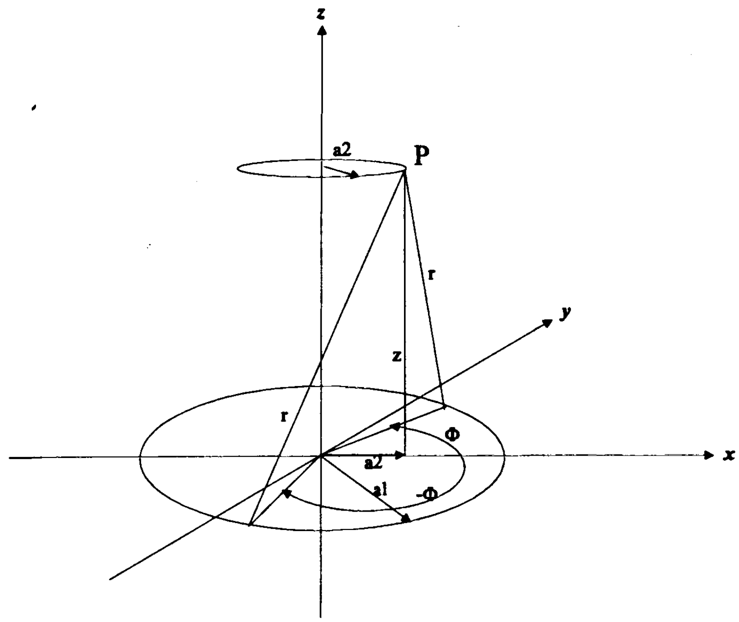


Fig.1.1
Two Coaxial Loops of Radii a1 and a2

In the special case of Fig.1 the only component of the vector potential ,A_Φ , at the point P on loop b due to a current in loop a is independent of Φ. Here z and a1 are constant because of the coaxial location of the two loops. It is also assumed that the circuit, i.e. the current conductor, is small compared to the dimension r or equivalently the current can be considered to be uniform and therefor have an equivalent line location at the radius a1. Under these conditions the vector potential has only the component A_Φ and is given by:

(3)
$$A_\Phi = \frac{\mu I}{2\pi}\int_0^{\pi}\frac{a1\cos(\phi)d\phi}{\sqrt{a1^2+a2^2+z^2-2a1a2\cos(\phi)}}$$

Since A_Φ is the only component and is everywhere tangent to the loop radius a2, then by equation (2) the mutual inductance M_{12} between two loops of radii a1 and a2 axially spaced at z_{12} is:

(4)
$$\boxed{M_{12} = \mu a1a2\int_0^{\pi}\frac{\cos(\phi)d\phi}{\sqrt{a1^2+a2^2+z_{12}^{\ 2}-2a1a2\cos(\phi)}}}$$

Equation (4) can be rearranged and expressed in terms of complete elliptic integrals. However, for design purposes it is far more convenient to let MATHCAD, or some similar program, evaluate the integrals as required.

1.2 Self inductance of a loop of finite size wire. Fig.2 is the diagram of a loop of radius b made of wire with radius a and permeability μ' carrying a uniformly distributed total current I. The total self inductance can be determined by evaluating the B field energy within the wire plus the flux coupled to the inside diameter of the loop. The B field inside the wire is a function of the radius and is given by:

(5) $$B(r) = \frac{\mu' r I}{2\pi a^2}$$

The total inductive energy stored in the wire is given by:

(6) $$W_i = \frac{1}{2\mu'}\int_V B^2 dV = \frac{\mu' I^2 b}{8\pi^2 a^4}\int_0^a 2\pi r^3 dr = \frac{1}{2}L_{11}I^2$$

The inductance component due to the energy inside the wire is:

(7) $$L_{11}' = \mu' \frac{b}{a}$$

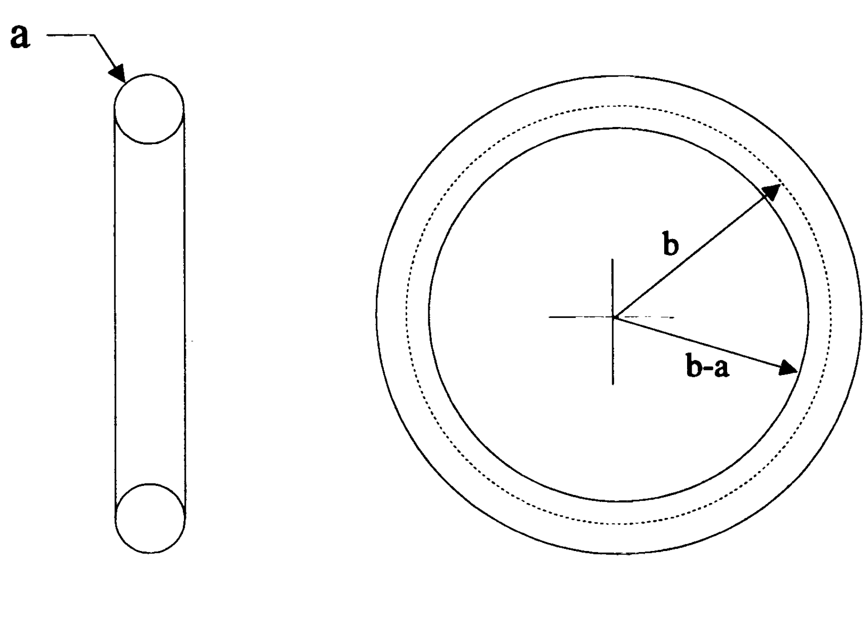


Fig. 1.2

Loop of radius b of a conductor of radius a

The total flux external to the conductor, i.e., the total flux contained within the radius b-a, is a measure of the inductance component external to the conductor. This flux is evaluated by equation (3). The vector potential component A_Φ evaluated at radius b-a and z=0 and multiplied by the circumference, 2π(b-a), is the external flux by equation (2). Therefore the total inductance of the loop is given by:

$$Lo = \mu' \frac{b}{4} + (\mu b(b-a)) \int_0^\pi \frac{\cos(\phi)d\phi}{\sqrt{b^2 + (b-a)^2 - 2b(b-a)\cos(\phi)}} \tag{8}$$

2. **<u>General Configuration of a Coaxial Coil</u>.** The general configuration of a coaxial coil is shown in Fig.2.1. The coil consists of N total coaxial loops, each having a specified wire radius, a_n, a loop mean radius, b_n, and an axial position z_n. The axial position is only important in the determination of the relative axial position's of the N loops, and therefore the reference for z is not important; however it is usually convenient to use the left most loop as the origin.

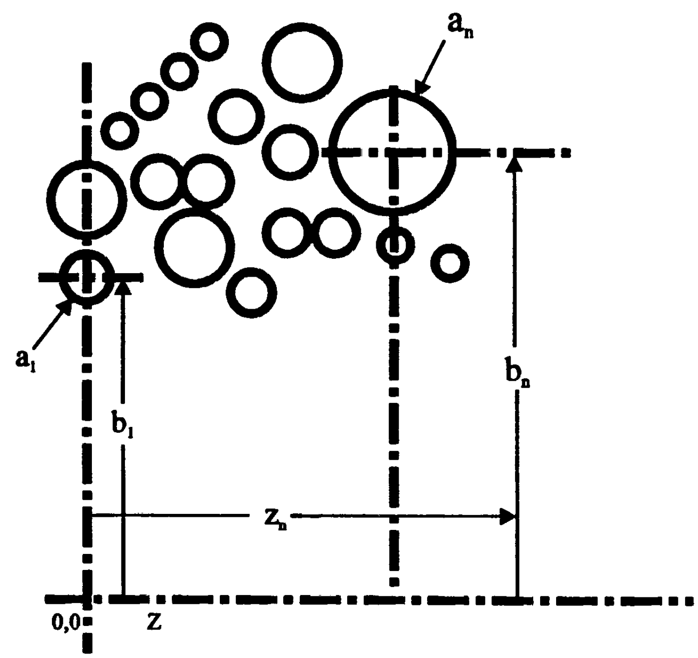


Fig. 2.1

General Configuration of a Coaxial Loop Coil

The basic definition of inductance referred to two terminals is expressed as:

(9) $$L_{1,2} = \frac{E_{1,2}}{\frac{di}{dt}}$$

Where: $E_{1,2}$ is the total voltage at terminals 1-2 as a result of the unit time rate of change of current passing through the terminals.

If we consider the general coil of Fig.2.1, the total voltage per unit time rate of change of current consists of the total self-inductance's of the loops plus the sum of all of the mutual inductances. Since $M_{v,w} = M_{w,v}$ each mutual will appear twice in determining the total voltage. Thus the total inductance of the coil is the sum of all of the elements in the inductance matrix as in equation (10). The diagonal elements are the self-inductances of the loops as determined by equation (8) and the symmetrical mutual inductances are determined by equation (4).

(10) $$\sum_{1}^{N}\sum_{1}^{N} \begin{matrix} L_1 & M_{1,2} & M_{1,n} & M_{1,N} \\ M_{1,2} & L_2 & M_{2,n} & M_{2,N} \\ M_{1,n} & M_{2,n} & L_n & M_{n,N} \\ M_{1,N} & M_{2,N} & M_{n,N} & L_N \end{matrix}$$

3. <u>Examples of Some Special Cases.</u>

3.1 Uniform Single Layer Solenoid. The simplest coil is a single layer solenoid with a constant diameter, pitch and wire size. We take the total number of turns, N=25; the wire radius, a=.001; the turn radius, b=.016; and the constant spacing between turns as .003. If we take the first turn as the origin the axial position of each turn becomes, z_n=0.003*(n-1). The MATHCAD formulation and calculations are given in Appendix A. The result is an inductance of 6.7913 microhenerys. This evaluation of inductance neglects the "cork-screw" nature of the coil and the magnetic vector potential; and is therefore accurate only for small a small pitch coil.

3.2 Non-Uniform Single Layer Solenoid. This example has a single layer; however, the wire size, loop radius, and turn to turn spacing are all taken to be a function of the turn number, N=44. The wire radius is defined as a_n = .001 + .0002*(n-1). The loop or turn radius is taken to vary linearly from .05 to .15 as, b_n = 0.05 + 0.00232558*(1-n). The turn position, z_n, is taken to vary as the square root of the turn number as, z_n = .0025*(n-1)^.5. The MATHCAD formulation and calculations are given in Appendix B. The result is an inductance of 378.122 microhenrys.

3.3 Inductance of a Coil With Rectangular Cross-section. The winding cross-section is rectangular. The wire radius is a. There are NTpL turns per layer and NL layers. Thus the total number of turns N, is equal to the product of NTpL and NL. The mean radius of the inner most layer is Radi. The center-center spacing between turns in a layer is Δh and the center to center spacing between turns in adjacent layers is Δv. The MATHCAD formulation and calculations are given is Appendix C.

3.4 Inductance of a the General Case Coil Composed of Series Connected Coaxial Turns. The general case of an inductance of a coil of series connected coaxial loops with the loops being of different wire sizes, different radii, and located in various z positions, is shown in Fig.2.1. The procedure for evaluating the inductance is no different in principle that the previous cases. One need only specify the radii of the wires, a_n; the radii of the loops, b_n; and the positions of the loops on the axis, z_n. If these array values cannot be generated by mathematical expressions they can be assigned as individual array elements.

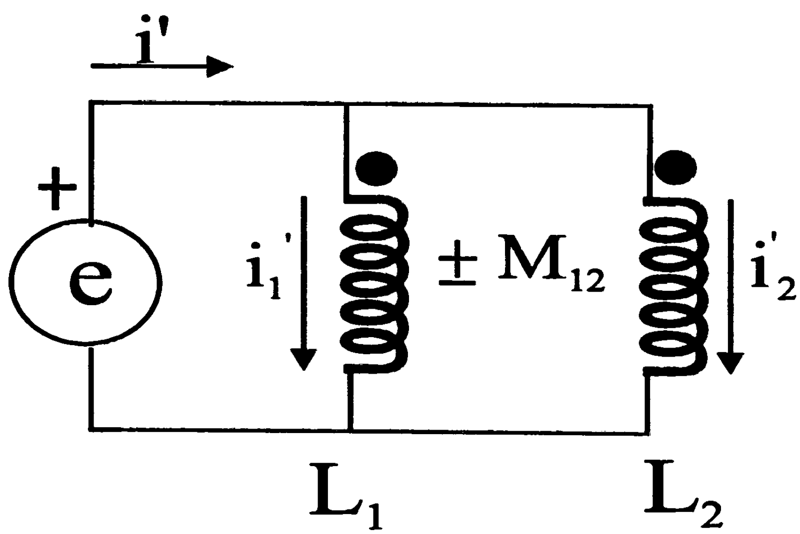
3.5 Inductance of Parallel Connected Loops . When two loops (or inductances) are connected in parallel the resulting equivalent inductance may be evaluated as follows. The circuit of the two loops in parallel is shown in Fig.3.5.1. The first derivatives of the branch currents can be determined using Creamer's rule as:

(11) $$\dot{i}_1 = \frac{e(L_2 \mp M_{12})}{L_1 L_2 \mp M_{12}^2} \qquad \dot{i}_2 = \frac{e(L_1 \mp M_{12})}{L_1 L_2 \mp M_{12}^2}$$

The equivalent inductance of the parallel connection, L_p, , is determined as:

(12) $$L_p = \frac{e}{\dot{i}_1 + \dot{i}_2} = \frac{L_1 L_2 \mp M_{12}^2}{L_1 + L_2 \mp 2M_{12}}$$

Fig.3.5.1
Circuit of Two Parallel Loops or Inductances



APPENDIX A

Inductance Calculation of a Uniform Solenoid $\mu := 4\cdot\pi\cdot 10^{-7}$

Number of Turns $N := 25$ Radius of Turn $b := .016$

Radius of Wire $a := .001$ Turn-turn Spacing $d := .003$

Self-Inductance of each Loop
$$Lo := \mu\cdot\left[\frac{b}{4} + b\cdot(b-a)\cdot\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{b^2+(b-a)^2-2\cdot b\cdot(b-a)\cdot\cos(\phi)}}\,d\phi\right]$$

$n := 1..N$

Axial position of each turn $z_n := d\cdot(n-1)$

$v := 1..N$ $w := 1..N$

Axial distance between v and w turns $\delta_{v,w} := \left|z_v - z_w\right|$

Mutual Inductance between v and w turns
$$M_{v,w} := (v \neq w)\cdot\mu\cdot b^2\cdot\left[\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{2\cdot b^2+\left(\delta_{v,w}\right)^2-2\cdot b^2\cdot\cos(\phi)}}\,d\phi\right]$$

$j := 1..N$

$M_{j,j} := Lo$

$k := 1..N$

Solenoid Diameter $Dia := 2\cdot b$ Solenoid Length $Zn := z_N$

$Dia = 0.032$ $Zn = 0.072$

Total Inductance
$$Ltot := \sum_{j=1}^{N}\sum_{k=1}^{N} M_{j,k}$$

$Ltot = 6.7913\cdot 10^{-6}$

APPENDIX B

Inductance calculation of a single layer non-uniform pitch solenoid $\mu := 4\cdot\pi\cdot 10^{-7}$

Number of Turns $N := 44$

$n := 1 .. N$

Wire Radii as a function of turn number $a_n := .001 + .0002\cdot(n-1)$

Loop or turn radii as a function of turn number $b_n := 0.05 + .00232558\cdot(n-1)$

Axial position of turn from first turn $z_n := .0025\cdot(n-1)^{0.5}$

$v := 1 .. N$ $w := 1 .. N$

Axial distance between v and w turns $\delta_{v,w} := \left|z_v - z_w\right|$

Self inductance of each turn

$$L_n := \mu\cdot\left[\frac{b_n}{4} + b_n\cdot\left(b_n - a_n\right)\cdot\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{\left(b_n\right)^2 + \left(b_n - a_n\right)^2 - 2\cdot b_n\cdot\left(b_n - a_n\right)\cdot\cos(\phi)}}\,d\phi\right]$$

Mutual Inductance between v and w turns

$j := 1 .. N$

$$M_{v,w} := (v \neq w)\cdot\mu\cdot b_v\cdot b_w\cdot\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{\left(b_v\right)^2 + \left(b_w\right)^2 + \left(\delta_{v,w}\right)^2 - 2\cdot b_v\cdot b_w\cdot\cos(\phi)}}\,d\phi$$

$M_{j,j} := L_j$

$bL := b_1$ $bR := b_N$

$k := 1 .. N$

$$zN := \left(\sum_{q=1}^{N} z_q\right)$$

Left End Radius $bL = 0.05$ Right End Radiuis $bR = 0.15$ Length $zN = 0.47764$

Total Inductance $Ltot := \sum_{j=1}^{N}\sum_{k=1}^{N} M_{j,k}$ $Ltot = 3.78122\cdot 10^{-4}$

APPENDIX C

Inductance Calculation of Coil with Rectangular Cross-section $\mu := 4\cdot\pi\cdot 10^{-7}$

Wire radius $a := .002$ Turn Spacing center-center in layers $\Delta h := 2\cdot a + .002$

Number of Turns per Layer $NTpL := 20$ Number of Layers $NL := 5$

Number of Turns $N := NTpL\cdot NL$ Turn Spacing center-center between layers $\Delta v := 2\cdot a + .005$

The axial coil length is $Lax := (NTpL)\cdot 2\cdot a$ Mean Radius of Inner Layer $Radi := .07$

$ID := 2\cdot Radi - \Delta v$

$n := 1 .. N$

The radius of a turn as a function of the turn number is $b_n := Radi + \Delta v\cdot\left(\mathrm{floor}\left(\frac{n-1}{NTpL}\right)\right)$

$OD := 2\cdot b_N + \Delta v$

The axial position of the turns as a function of the turn number is $z_n := \Delta h\cdot \mathrm{mod}(n-1, NTpL)$

$v := 1 .. N$ $w := 1 .. N$

$Lax := z_N + \Delta h$

The axial distance between turns v,w is $\delta_{v,w} := \left|z_v - z_w\right|$

The self-inductance of each turn is

$$L_n := \mu\cdot\left[\frac{b_n}{4} + b_n\cdot(b_n - a)\cdot\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{(b_n)^2 + (b_n - 1)^2 - 2\cdot b_n\cdot(b_n - a)\cdot\cos(\phi)}}\,d\phi\right]$$

The Mutual Inductance between turns v, and w is

$$M_{v,w} := (v\neq w)\cdot\mu\cdot b_v\cdot b_w\cdot\int_0^{\pi}\frac{\cos(\phi)}{\sqrt{(b_v)^2 + (b_w)^2 + (\delta_{v,w})^2 - 2\cdot b_v\cdot b_w\cdot\cos(\phi)}}\,d\phi$$

$j := 1 .. N$

$M_{j,j} := L_j$

$k := 1 .. N$

$$Ltot := \sum_{j=1}^{N}\sum_{k=1}^{N} M_{j,k}$$

Total Inductance is $Ltot = 0.00116$

Inner Diameter $ID = 0.131$ Outer Diameter $OD = 0.221$ Length $Lax = 0.12$