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Note 39

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Matching Modulated Electron Beam to Waveguide

Carl E. Baum

Weapons Laboratory

Abstract

A classical problem in generating electromagnetic fields in a closed perfectly conducting waveguide involves passing an electron beam modulated at some particular frequency through the guide. This paper discusses this problem in the context of an idealized sheet beam for which one can set up a simple boundary value problem for the $H_{1,0}$ mode in a conventional rectangular waveguide.

I. Introduction

One of the classical problems of microwave engineering is that of converting a modulated electron beam into a sinusoidal electromagnetic wave in or on some waveguiding structure. This is formulated in terms of integrals over the current weighted by the appropriate waveguide mode [3].

This paper considers a canonical problem which conceptually simplifies the problem. In a conventional rectangular waveguide the electron beam is chosen so as to match the $H_{1,0}$ mode (lowest order mode) in a form which allows the solution of a simple boundary-value problem. This gives some simple formulas for the modal amplitude. Noting the linear approximation concerning the electron beam one can estimate limitations on the modal amplitude, consistent with the beam parameters.

II. Canonical Form of Sheet Beam Traversing Waveguide

As illustrated for two cases in Figure 2.1, let there be a sheet beam on the z=0 plane traversing rectangular waveguides with surface current density (idealized) as

$$\vec{J}_s(x,y,t) = I_o \frac{\pi}{2a} \sin\left(\frac{\pi x}{a}\right) \left[1 + \nu \cos\left(\omega_o t\right)\right] \vec{1}_y$$

$$I_o = \text{average total current (average over time)} \tag{2.1}$$

$$0 \le \nu \le 1$$

where ν can be regarded as some kind of modulation efficiency associated with the lowest H mode ($H_{1,0}$ mode) and radian frequency ω_o. This canonical form is chosen for its ease in matching to the $H_{1,0}$ mode, but it has limitations. For example, it does not allow for the transit time of the electrons across the height b of the guide, but if b is small compared to a this will not be a severe limitation.

More generally one can define a sheet beam in the form

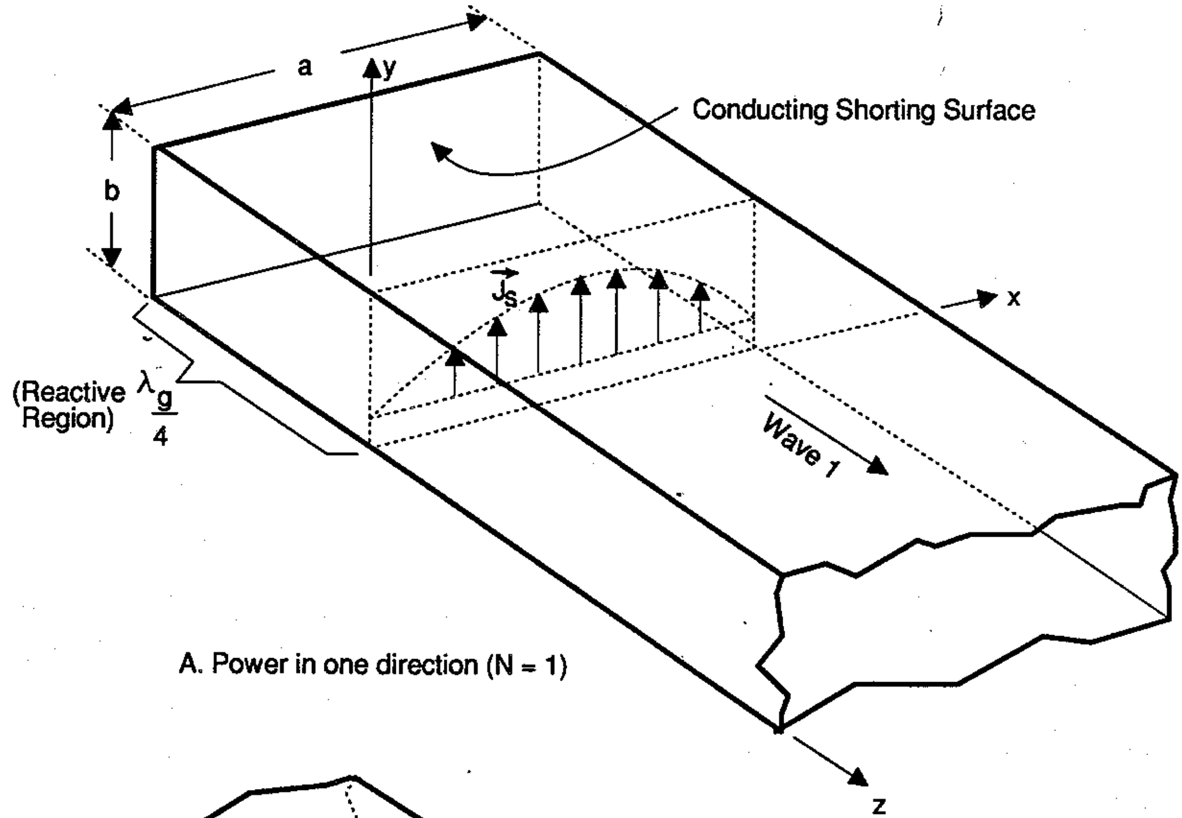
$$\vec{J}_s(x,y,t) = I_o \vec{f}(x,y,t)$$

$$\vec{1}_z \cdot \vec{f}(x,y,t) = 0$$

$$T = \frac{1}{f_o} = \frac{2\pi}{\omega_o} \equiv \text{period of } \vec{f} \tag{2.2}$$

$$\int_0^a \frac{1}{T} \int_0^T \vec{1}_y \cdot \vec{f}(x,y,t)\, dt\, dx = 1 \text{ (normalization)}$$

In this form $\vec{J}_s$ can contain harmonics of f_o, a more realistic situation. A special form which still allows for electron transit across the guide is



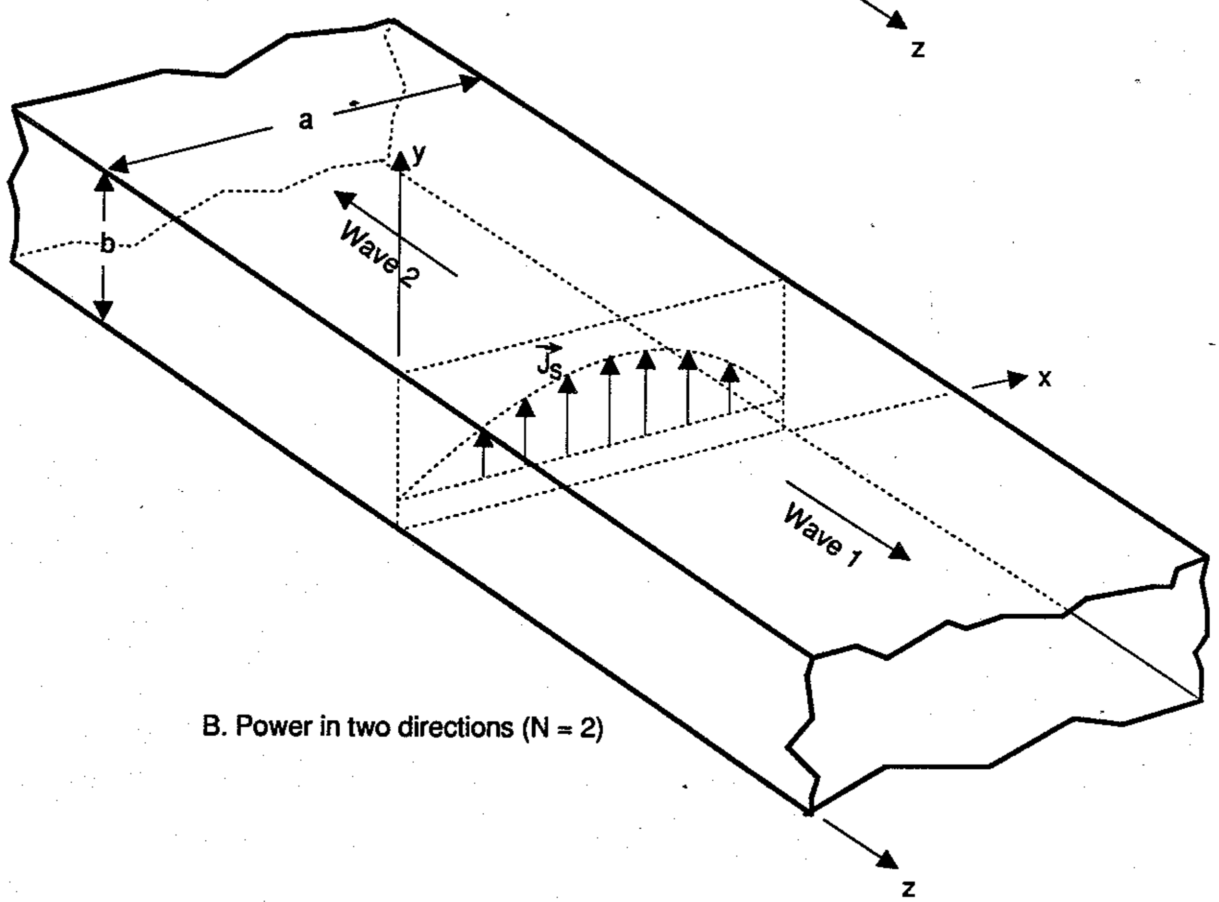


Figure 2.1. Electron Beam Traversing Rectangular Waveguide

$$\vec{f}(x,y,t) = f(x)\ g\left(t - \frac{y}{v}\right)\vec{1}_y$$

$0 < |v| < c$ (v can be + or -)

$$\int_0^a f(x)\,dx = 1 \tag{2.3}$$

$$\frac{1}{T}\int_0^T g\left(t - \frac{y}{v}\right) dt = 1$$

Relating this to the form in (2.1) one can think of ν as representing the component of the actual distribution at frequency f_o, distributed uniformly in y, and as $\sin(\frac{\pi x}{a})$ in x.

Using the form in (2.1) one can integrate over a period and the guide cross section as

$$X_o \equiv \frac{1}{Tb}\int_0^T\int_0^b\int_0^a \vec{J}_s(x,y,t)\cdot\vec{1}_y\ \sin\left(\frac{\pi x}{a}\right)\cos\left(\omega_o t\right)\ dx\ dy\ dt = I_o\ \frac{\pi}{8}\ \nu \tag{2.4}$$

Similarly using the form in (2.2) gives a more general expression

$$X = I_o\ \frac{1}{Tb}\int_0^T\int_0^b\int_0^a \vec{f}(x,y,t)\cdot\vec{1}_y\ \sin\left(\frac{\pi x}{a}\right)\cos\left(\omega_o t\right)\ dx\ dy\ dt \tag{2.5}$$

with appropriate attention to the phase in $\vec{f}$ or with a phase shift added to $\omega_o t$ to maximize the resulting magnitude. Comparing these forms allows one to compare the efficiencies of various modulated beams for exciting the $H_{1,0}$ mode.

Note that this still neglects some phenomena, such as beam spreading in the z direction, but this could also be included with appropriate weight from the electric field in the $H_{1,0}$ mode.

Assuming the special form for $\vec{f}$ as in (2.3) we have

$$X = I_o \left\{ \int_0^a f(x) \sin\left(\frac{\pi x}{a}\right) dx \right\} \left\{ \frac{1}{b} \int_0^b \frac{1}{T} \int_0^T g\left(t - \frac{y}{v}\right) \cos\left(\omega_o t\right) dt \, dy \right\} \tag{2.6}$$

Defining an average over y of the time function as

$$G(t) \equiv \frac{1}{b} \int_0^b g\left(t - \frac{y}{v}\right) dy \tag{2.7}$$

gives

$$X = I_o \left\{ \int_0^a f(x) \sin\left(\frac{\pi x}{a}\right) dx \right\} \left\{ \frac{1}{T} \int_0^T G(t) \cos\left(\omega_o t\right) dt \right\} \tag{2.8}$$

as a more general form for the modulation efficiency. Again note the definition of phase (or t=0) to maximize the integral with respect to the $\cos(\omega_o t)$ weight (or null with respect to a $\sin(\omega_o t)$ weight).

For another shape of beam let us suppose

$$f(x) \equiv \frac{\pi}{2a} \sin\left(\frac{\pi x}{a}\right)$$

$$G(t) \equiv \begin{cases} 2 \text{ for } \cos\left(\omega_o t\right) > 0 \\ 0 \text{ for } \cos\left(\omega_o t\right) < 0 \end{cases} \tag{2.9}$$

where b is assumed small enough that electron velocities can be assumed infinite. This square-wave modulation gives

$$X = I_o \frac{\pi}{4} \frac{1}{T} \int_0^T G(t) \cos\left(\omega_o t\right) dt$$

$$= I_o \frac{\pi}{4} \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} \cos\left(\omega_o t\right) dt$$

$$= I_o \frac{\pi}{4} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\omega_o t\right) d\left(\omega_o t\right)$$

$$= \frac{I_o}{2} \tag{2.10}$$

which is a slight increase over (2.4) for $\nu=1$.

Another idealized beam is formed by bunching the beam in the center of the guide ($x=a/2$) for which

$$f(x) = \delta\left(x - \frac{a}{2}\right)$$

$$\int_0^a f(x) \sin\left(\frac{\pi x}{a}\right) dx = 1 \tag{2.11}$$

Then still using the square-wave form of G(t) we have

$$X = I_o \frac{1}{T} \int_0^T G(t) \cos\left(\omega_o t\right) dt$$

$$= \frac{2}{\pi} I_o \tag{2.12}$$

which is a small increase again.

Comparing these various results one can then use the form in (2.1) to obtain estimates of the $H_{1,0}$ excitation, noting that other interesting forms of beam modulation give similar answers.

III. Matching to $H_{1,0}$ Mode

The various modes of a closed perfectly conducting waveguide are a classical problem in electromagnetic theory which are discussed in numerous places (e.g. [1,3]). For a rectangular waveguide, these are especially well known. The lowest order H mode is the $H_{1,0}$ mode described for radian frequency ω_o and propagation in the +z direction by [1].

$$\vec{E}_t = -Z_h H_o \sin\left(\frac{\pi x}{a}\right) \vec{1}_y e^{j\omega_o t - \gamma_z z}$$

$$\vec{H}_t = H_o \sin\left(\frac{\pi x}{a}\right) \vec{1}_x e^{j\omega_o t - \gamma_z z}$$

$$H_z = H_o \frac{K_h}{jK\zeta_h} \cos\left(\frac{\pi x}{a}\right) e^{j\omega_o t - \gamma_z z} \tag{3.1}$$

$$= -j \frac{H_o}{\zeta_h} \frac{\lambda_h}{\lambda_o} \cos\left(\frac{\pi x}{a}\right) e^{j\omega_o t - \gamma_z z}$$

where

$$\gamma_z = jk_z = jk\zeta_h \ , \ \lambda_g = \lambda_o \zeta_h^{-1} \text{ (guide wavelength)}$$

$$\zeta_h = \left[1 - \left(\frac{\lambda_o}{\lambda_h}\right)^2\right]^{1/2} = \left[1 - \left(\frac{\lambda_o}{2a}\right)^2\right]^{1/2} \tag{3.2}$$

$$k = \frac{\omega_o}{c} \ , \ \omega_o = 2\pi f_o \ , \ \lambda_o = \frac{c}{f_o} = \frac{2\pi c}{\omega_o}$$

$$\lambda_h = 2a \text{ (cutoff wavelength)}, \ k_h = \frac{2\pi}{\lambda_h} = \frac{\pi}{a}$$

$$Z_h = Z_o \zeta_h^{-1} \text{ (modal impedance)}$$

$$Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \simeq 377\Omega \text{ (free space wave impedance)}$$

$$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \simeq 3 \times 10^8 \frac{m}{s} \text{ (speed of light)}$$

where H_o is some scaling constant to be computed. Note that the frequency is chosen as ω_o per the previous section (and likewise for f_o and λ_o). Appropriate change of signs gives propagation in the -z direction. Here (3.1) gives the complex form; for present purposes we need only consider the real part when matching to the beam.

As discussed in [1] for optimum power-handling capability one should operate with the frequency just below the cutoff frequency of the next H mode which has a wavelength of the larger of a and 2b. Then in the limit we can take

$$\lambda_o = \frac{1}{2}\lambda_h = a \ , \ 0 < b \leq \frac{a}{2}$$

$$\zeta_h = \frac{\sqrt{3}}{2} \simeq .866 \qquad (3.3)$$

$$Z_h = Z_o \, \zeta_h^{-1} \simeq 435\Omega$$

recognizing that the actual wavelength is just a little longer than this. Note that with the restriction on b we still have some flexibility of choosing b to match beam parameters.

Now referring back to Figure 2.1 let us match this $H_{1,0}$ mode to the electron beam. In Figure 2.1A there is a quarter wavelength section of waveguide terminated in a perfectly conducting surface (short circuit) with the guide wavelength given by

$$\lambda_g = \lambda_o \, \zeta_h^{-1} \qquad (3.4)$$

In this case the magnetic field tangential to and just behind the z=0 plane (z=0-) in the $H_{1,0}$ mode is zero. The boundary condition at z=0 is

$$\vec{J}_s(x,y,t) = \vec{1}_z \times \left[\vec{H}(x,y,z,t)\Big|_{z=0+} - \vec{H}(x,y,z,t)\Big|_{z=0-}\right] \qquad (3.5)$$

Applying this to the $H_{1,0}$ mode (real part from (3.1)) with radian frequency ω_o and the form of the surface current density in (2.1) we have

$$H_o = I_o \frac{\pi}{2a} \nu \qquad (3.6)$$

As indicated in the figure, there is one wave propagating away from the beam in the +z direction. Let us designate this case by N=1.

Another case is given in Figure 2.1B which we designate by N=2. Here there are two waves propagating away from the beam in the +z and -z directions. Noting from symmetry that, except for some signs, the two waves are equal, and applying (3.5) we have

$$2H_o = I_o \frac{\pi}{2a} \nu \tag{3.7}$$

Summarizing both cases we have

$$H_o = \frac{I_o}{N} \frac{\pi}{2a} \nu \tag{3.8}$$

In this form we can consider both configurations in Figure 2.1 in the same formulas.

Having the magnetic field in the $H_{1,0}$ mode one also has the peak electric field as

$$E_o = Z_h H_o = Z_h \frac{I_o}{N} \frac{\pi}{2a} \nu \tag{3.9}$$

One can also then define a peak "voltage" associated with the $H_{1,0}$ mode as

$$V_o = -\int_0^b \vec{E}_t \Big|_{x=\frac{a}{2}} \cdot \vec{1}_y \, dy = E_o b = Z_h H_o b = Z_h \frac{I_o}{N} \frac{\pi b}{2a} \nu \tag{3.10}$$

The power in the waveguide ($H_{1,0}$ mode) is

$$\begin{aligned} P_w &= N \int_0^b \int_0^a \left[\vec{E}_t \times \vec{H}_t\right]\Big|_{z=0+} \cdot \vec{1}_z \, dx \, dy \\ &= N E_o H_o \frac{ab}{2} \cos^2\left(\omega_o t\right) \end{aligned} \tag{3.11}$$

which is the usual average sense is

$$P_{w_{avg}} = \frac{1}{2} P_{w_{max}} = N \frac{ab}{4} E_o H_o = N \frac{ab}{4} Z_h H_o^2$$

$$= N \frac{ab}{4} \frac{E_o^2}{Z_h} = \frac{\pi}{8} \nu V_o I_o = \frac{1}{N} \frac{\pi^2}{16} \frac{b}{a} \nu^2 Z_h I_o^2 \qquad (3.12)$$

Letting V_e be the beam voltage, i.e., the positive potential through which the electrons have been accelerated before passing through the waveguide (or equivalently the electron energy in e.V.) the average power available in the beam is (with I_o taken positive)

$$P_{b_{avg}} = V_e I_o \qquad (3.13)$$

One can define an efficiency as

$$\eta = \frac{P_{w_{avg}}}{P_{b_{avg}}} = \frac{\pi}{8} \nu \frac{V_o}{V_e} = \frac{1}{N} \frac{\pi^2}{16} \frac{b}{a} \nu^2 Z_h \frac{I_o}{V_e} \qquad (3.14)$$

Note that other types of modulated beams can be considered by using (from (2.4) and (2.5)) an effective parameter for ν as

$$\nu_{eff} = \frac{8}{\pi} \frac{X}{I_o} \qquad (3.15)$$

with, for example, the special results as in (2.10) and (2.12). However, one needs to be cautious here as the efficiencies approach unity because the electrons give up energy to the field and slow down from their initial speeds, making the problem nonlinear.

There are other ways to look at the onset of nonlinearities, this being a limitation on the basic linear model. Compare V_o to V_e (both taken positive). For small b $\ni$ electron transit time across the guide can be neglected we can require

$$0 \leq \frac{V_o}{V_e} \leq 1 \qquad (3.16)$$

This merely states that during the peak retarding electric field the electrons can lose no more than their initial energy.

Another potential limitation concerns the space charge in the electron beam. Assuming an electron speed v, the sheet beam (assuming only a y component) has a surface charge density

$$\rho_s = \frac{J_{s_y}}{v} \quad \text{(negative)} \tag{3.17}$$

This gives a normal electric field (± z direction)

$$E_s = \left|\frac{\rho_s}{2\epsilon_o}\right| = \left|\frac{J_{s_y}}{2\epsilon_o v}\right| = \frac{1}{2}\,\frac{c}{|v|}\,Z_o\,\left|J_{s_y}\right| \tag{3.18}$$

Using the ideal form of $\vec{J}_s$ in (2.1) gives

$$E_s = \frac{1}{2}\,\frac{c}{|v|}\,Z_o\,\left|I_o\right|\,\frac{\pi}{2a}\,\sin\!\left(\frac{\pi x}{a}\right)\left[1 + \nu\,\cos\!\left(\omega_o t\right)\right] \tag{3.19}$$

with maximum value

$$E_{s_{max}} = \frac{c}{|v|}\,Z_o\,\left|I_o\right|\,\frac{\pi}{2a}\,\frac{1+\nu}{2} \tag{3.20}$$

Comparing this to the peak electric field for the $H_{1,0}$ mode in (3.9) we can see that they are comparable for ν near 1 and $|v|$ near c.

As one increases the efficiency of conversion η of average beam power to average $H_{1,0}$ mode power (i.e. as $\eta \to 1$) we then expect nonlinearities to become significant. In such a case the present formulas break down and much more detailed calculations are appropriate. Note that (3.14) can be rearranged (using the effective beam parameter in (3.15)) as

$$\frac{V_e}{I_o} = \frac{1}{\eta}\frac{\pi^2}{16} \quad \nu^2_{eff} \quad \frac{b}{a}\frac{Z_h}{N}$$

$$= \chi \frac{b}{a}\frac{Z_h}{N} \tag{3.21}$$

$$\chi \equiv \frac{1}{\eta}\frac{\pi^2}{16} \quad \nu^2_{eff} \quad = \frac{4}{\eta}\left[\frac{X}{I_o}\right]^2$$

where χ is of rough order 1 under optimum conditions (η and ν_{eff} of order 1). Note then that V_e/I_o (dimensionally a resistance for the electron beam) is of the order b/a times Z_h (the modal impedance) divided by N (the number of "parallel" waveguide outputs). For given beam parameters this gives some rough estimate of the optimum b/a for the waveguide.

IV. Transformation of Guide Height

The choice of a is related to the choice of ω_o, or equivalently λ_o, with

$$a < \lambda_o < 2a \tag{4.1}$$

but ideally near a [1] with

$$0 < b \leq \frac{a}{2} \tag{4.2}$$

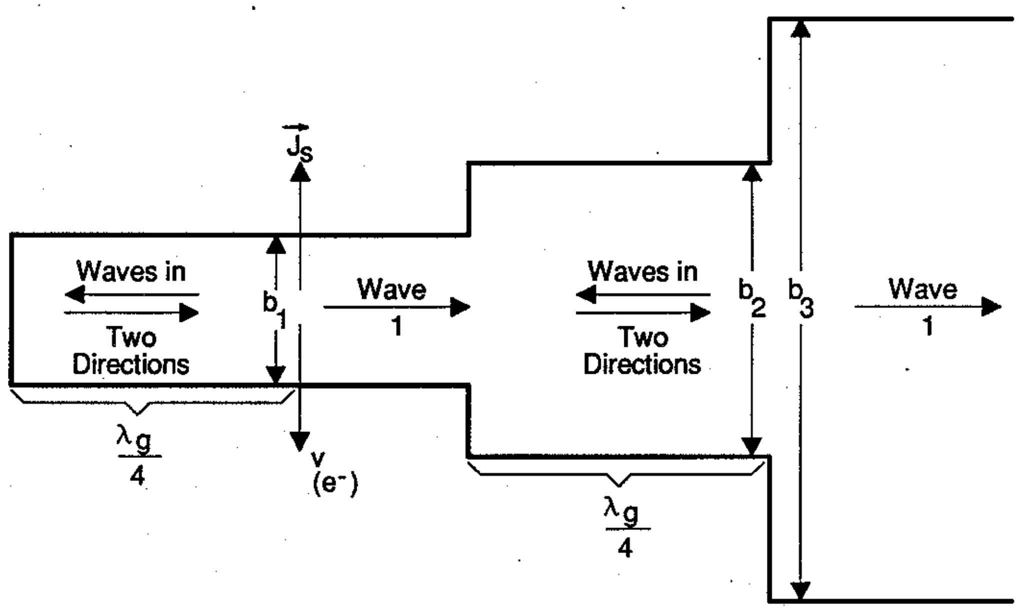
Within this last restriction, the guide wavelength is not a function of b. Efficiency of coupling the electron beam to the $H_{1,0}$ mode then influences the choice of b as in (3.21).

Assuming that the desired value of b is <<a/2 one may wish to change this value to some other, say a/2, the usual dimension of a rectangular waveguide. Other useful choices for this parameter lie in the range given by (4.2). As discussed in [1] perfectly conducting sheets can be placed on planes of constant y without perturbing the $H_{1,0}$ mode. This can be used to divide such a mode in one waveguide into $H_{1,0}$ modes on 2 or more waveguides, or combine $H_{1,0}$ modes from 2 or more waveguides into one mode in a single waveguide.

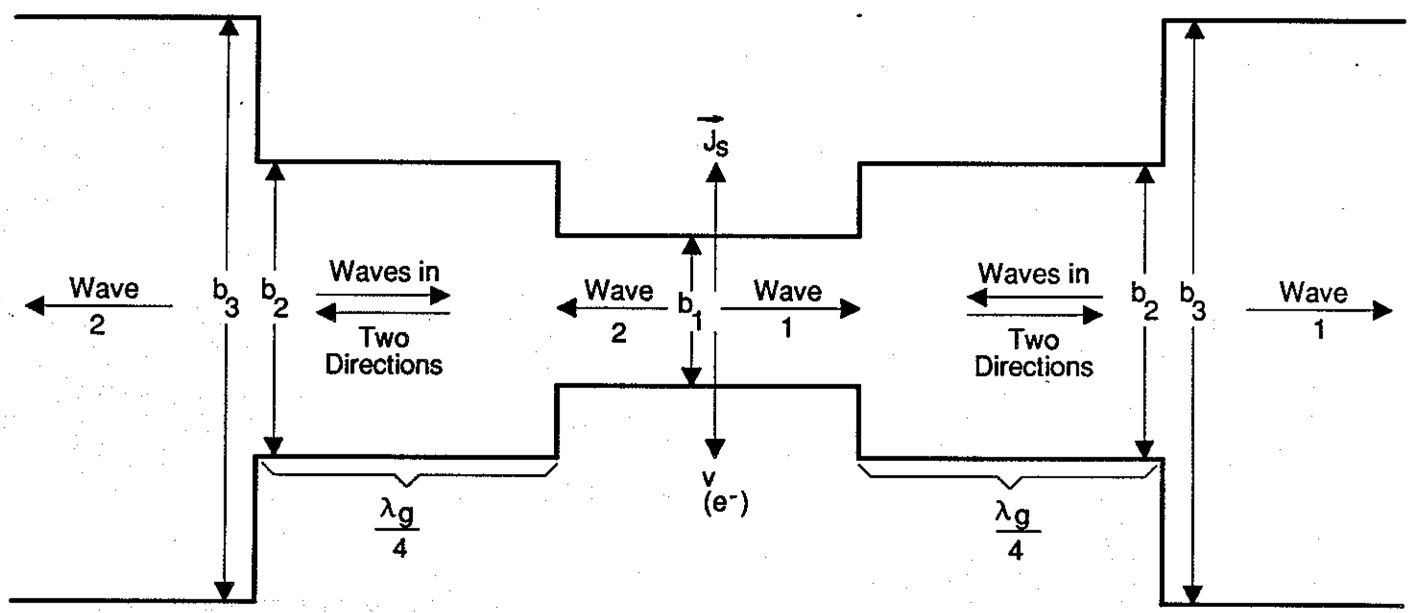
So starting with one height b_1 at the electron beam, the problem is to transform the waveguide height to some other value b_3. Such transformers come in many kinds [2,3]. A simple example is the quarter-wave transformer illustrated in Figure 4.1. The transformer section is a quarter guide wavelength long and has height b_2 given by

$$b_2 = \sqrt{b_1 \; b_3} \tag{4.3}$$

Note that due to the special properties of the $H_{1,0}$ mode, the width a can be the same for all sections and the guide wavelength is not a function of the height. However, the equivalent impedance of the guide is Z_h b/a (i.e. $(bE_t)/(aH_t)$) so that (4.3) is related to the requirement that the transformer section have a geometric mean impedance. Note that for various reasons (such as bandwidth) one may wish to use a more gradual type of transformer section.



A. Power in one direction (N = 1)



B. Power in two directions (N = 2)

Figure 4.1. Quarter-Wave Transformer in Rectangular Waveguide

V. Compensation for Finite Electron Speed

In Section 2 the electron beam traversing the waveguide on the z=0 plane is given a speed v, which is idealized as infinite for some purposes (as in (2.1)). Provided b is sufficiently small (<<a/2) and |v/c| is not too small (compared to 1) then this is an appropriate approximation. Still one may wish to compensate for the finite electron speed to improve the excitation efficiency for the $H_{1,0}$ mode.

As in [1] let us divide the guide at planes of constant y so that the transit time of the electrons across the spacing Δ between the conducting sheets is made as small as desired. Then in the waveguides of height Δ the electron transit-time is not significant and we can approximate in (2.7) as

$$G(t) \simeq g\left(t - \frac{y}{v}\right) \tag{5.1}$$

for each of these guides. The different guides will, however, have different phases on their $H_{1,0}$ modes (reflected in the y/v term in g).

Consider the quarter-wave section as in Figure 2.1A and Figure 4.1A with shorting surface to provide an open circuit (no tangential magnetic field) at z=0-. This concept of dividing the guide on planes of constant y is illustrated in Figure 5.1. Here the electron beam on the z=0 plane has electrons traveling in the -y direction (giving current in the +y direction). Note that the finite speed of the electrons means that the signal at frequency f_o is first induced in the uppermost subguide, and is progressively induced in the lower subguides. As illustrated in Figure 5.1, the wavefront (in a phase speed sense) has moved farther to the left in the upper subguides (and reflects back first in these). This provides an open circuit at z=0- at each subguide (for frequency f_o) with the phase of the electric field at the subguide openings matched to that of the electron beam propagating in the -y direction.

For a waveguide transporting power away from the electron beam, the guide can be subdivided as in Figure 5.2 so that the subguides are of different lengths. There are many different ways to do this. The example in Figure 5.2 shows one way involving an E-plane bend of the waveguide. Let this bend be described by

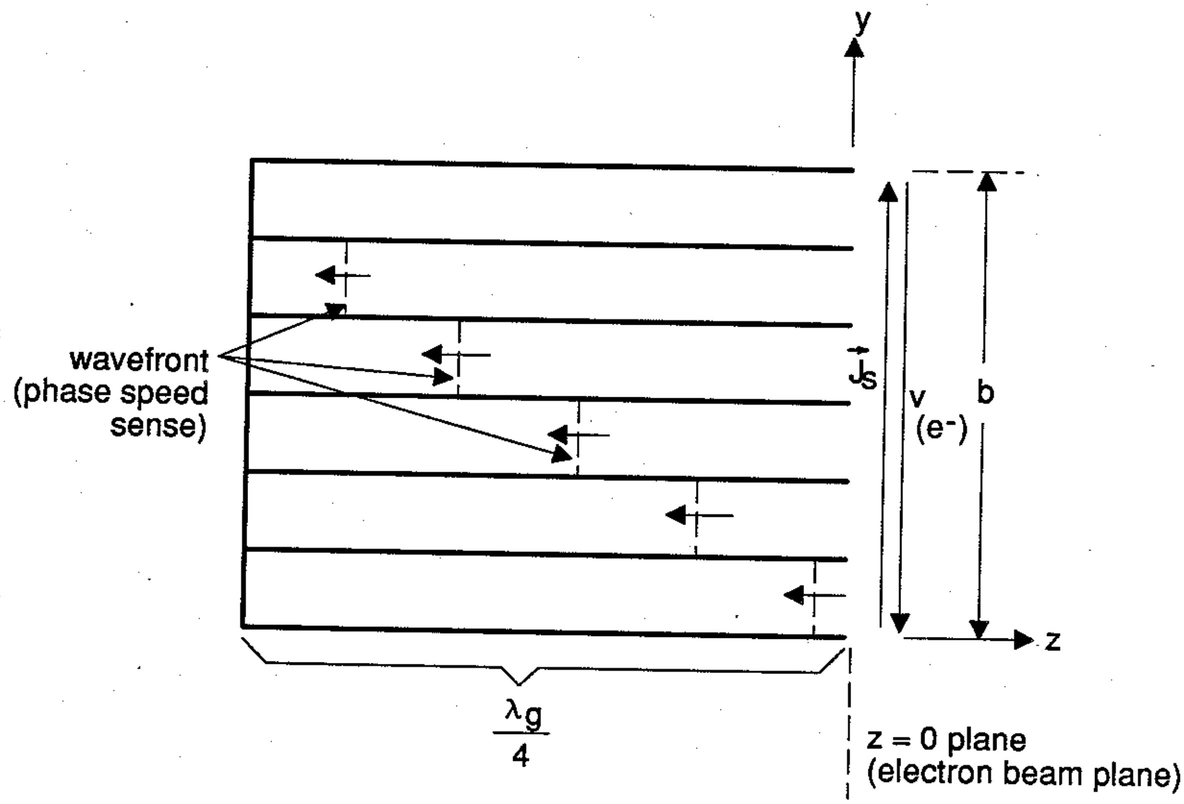


Figure 5.1. Partitioned Short-Circuit Rectangular Waveguide

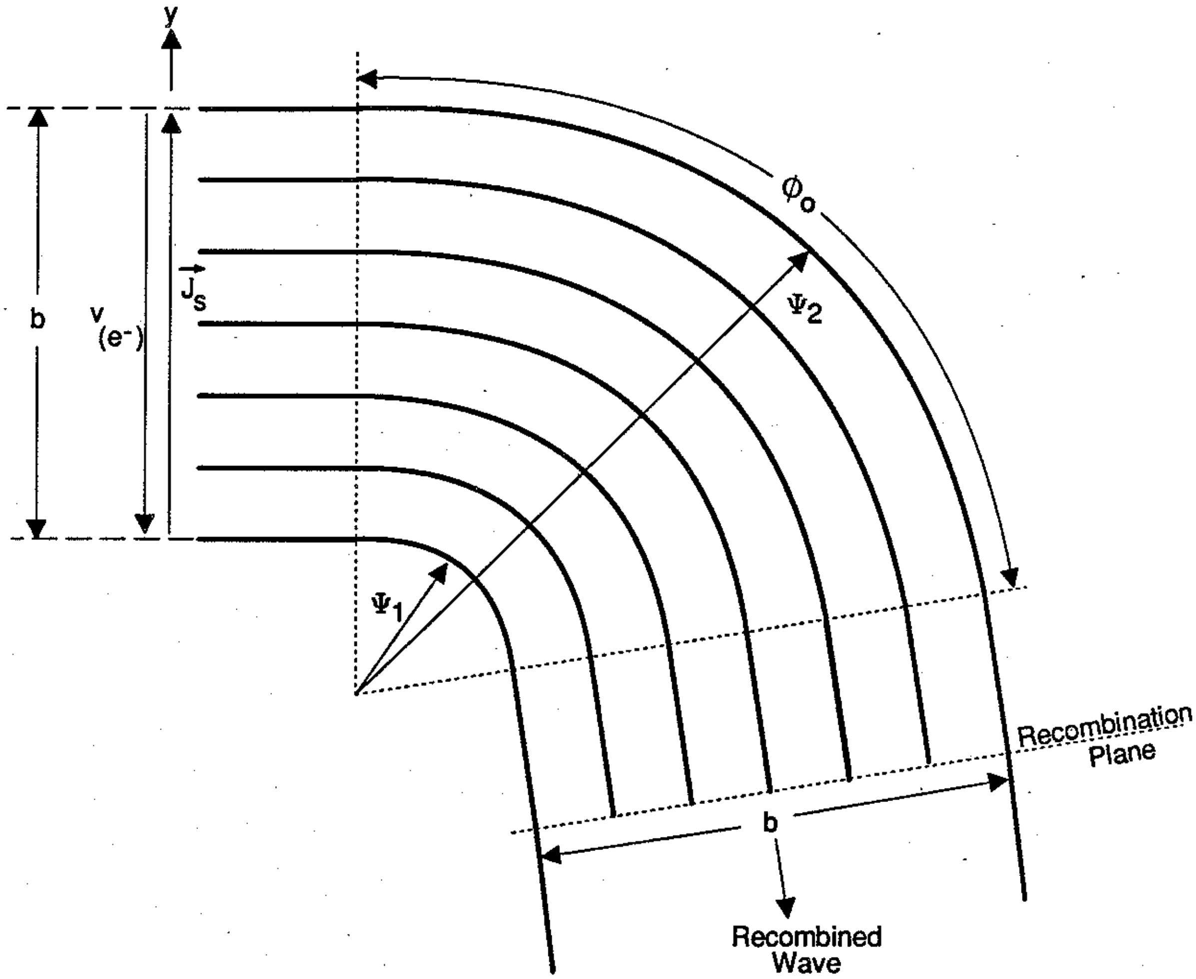


Figure 5.2. Partitioned Phase-Correcting Rectangular Waveguide

$\Psi_1 \equiv$ inner radius

$\Psi_2 \equiv$ outer radius

$\Psi_2 - \Psi_1 = b$

$\phi_o \equiv$ angle of bend (radians) (5.2)

Now the transit-time of the electrons across the guide is

$$T_e = \frac{b}{v} = \frac{b}{\beta c}$$

$$\beta = \frac{v}{c} = \left[1 - \left[\frac{V_m}{V_m + V_e}\right]^2\right]^{1/2} < 1 \qquad (5.3)$$

V_m = energy of electron at rest in potential units

$\approx .51$ MV

The additional transit time (in a phase-velocity sense) in the outer subguide (over the inner one) is just

$$T_g = \frac{\Psi_2 \ \phi_o}{v_g} - \frac{\Psi_1 \ \phi_o}{v_g} = \frac{b \ \phi_o}{v_g}$$

v_g = phase speed in waveguide (5.4)

$$\frac{v_g}{c} = \frac{\lambda_g}{\lambda_o} = \zeta_h^{-1} > 1$$

Equating these two times so that the waves are all in phase on the recombination plane (perpendicular to the new guide direction after the bend) gives

$$T_e = T_g = \frac{b}{\beta c} = \frac{b \ \phi_o}{v_g}$$

$$\phi_o = \frac{v_g}{\beta c} = [\beta \zeta_h]^{-1} > 1 \text{ radian} \qquad (5.5)$$

The angle of bend is rather large, and if this is undesirable other geometries can be used to lengthen the subguides first excited by the electron beam, including physical separation of the subguides (as discussed in [1]). One could also vary the width a of the subguides to appropriately increase the phase speeds (with appropriate attention to any impedance changes).

One also needs to integrate the electron beam with the waveguide structure. The beam should pass through the conducting walls without discontinuities which would significantly perturb the desired fields. A classical technique for doing this is to replace portions of the walls through which the beam is to pass by a conducting grid or grating as in Figure 5.3. Most of the beam then passes through the holes or separation between the conductors. One also needs to appropriately connect the beam source to one wall and a beam exit or dump to the opposite wall of the overall waveguide.

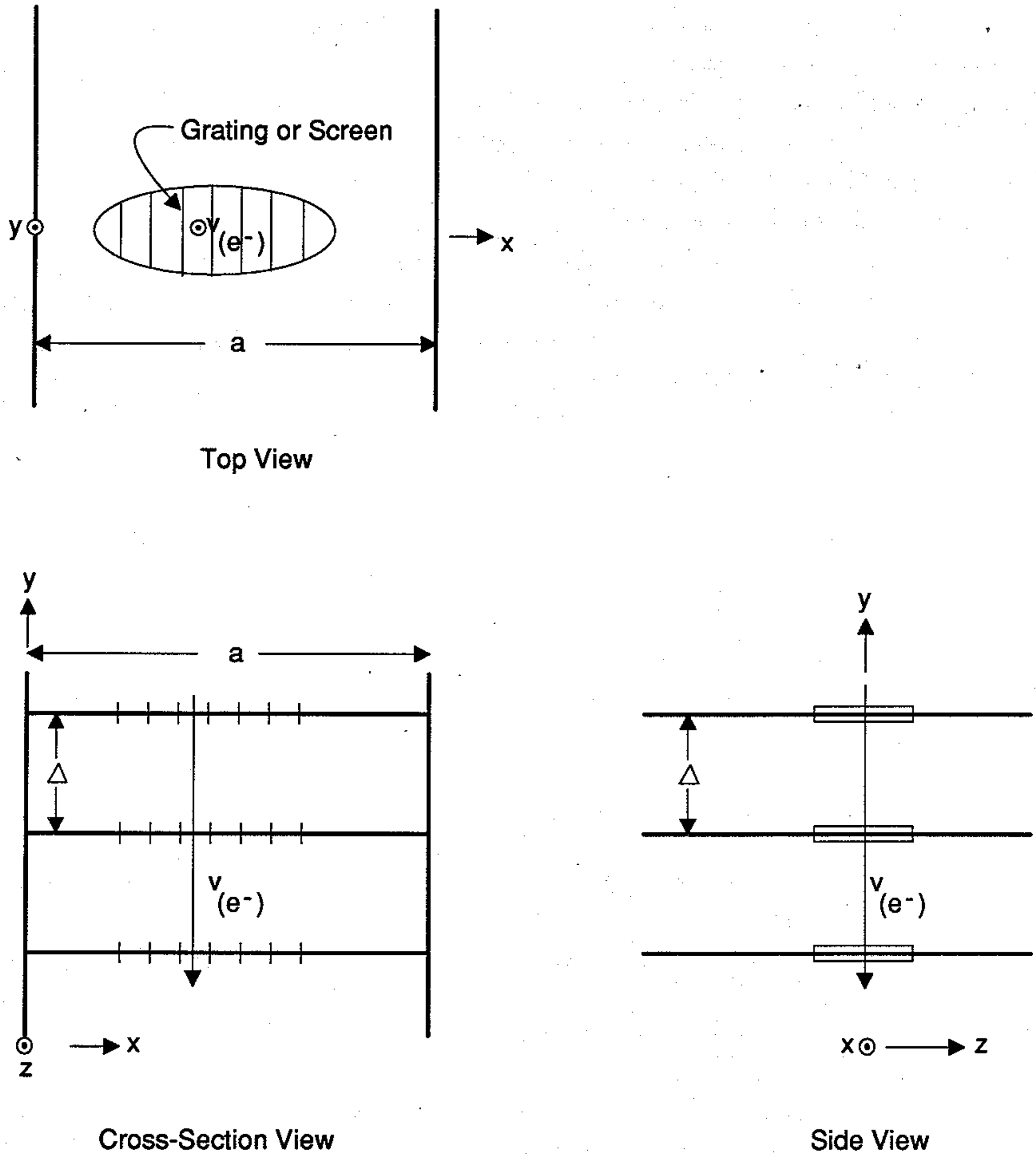


Figure 5.3. Passage of Electron Beam Through Subguide Walls

VI. Concluding Remarks

While the basic concepts are rather straightforward, there are various problems in optimizing the coupling of a modulated electron beam to a waveguide. The present analysis is linear and can be used to estimate various parameters. However, an efficient design will remove a significant portion of the beam energy, making the problem nonlinear for which a more detailed analysis and/or experiments are appropriate. One can "tune up" various configurations such as by varying the shorted quarter-wave section (for $N=1$) to present a slightly reactive impedance at the electron beam to try to compensate for various non-ideal properties of the beam. Transformer sections of waveguide can also be varied to optimize the performance. In an experimental configuration one can try to make various dimensions (such as b_1) variable to determine best operating conditions.

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