CESDN E3

APPROXIMATE CALCULATION OF ENERGY TRANSFER THROUGH AN EXPONENTIAL LINE

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The exponential line is a type of variable impedance transmission line which can be used as a pulse transformer and for which the differential equation describing the spatial dependence of the wave propagation is easily soluble. The complete solution of the partial differential equation is then obtained by a further integration which gives the time dependence. This latter integration is equivalent to finding an inverse Laplace transform, and it can be difficult simply because tables of transforms are not extensive enough to contain the functions corresponding to many circuit combinations. Lewis and Wells describe an approximate theory for the exponential line which gives the time dependent waveform approximated by the constant and linear terms of its expansion as a power series in time. Such an approximation is clearly good only for small values of time. However Lewis and Wells reason that, to be useful as a transformer, an exponential line should not introduce serious waveform distortion, and hence this linear approximation is adequate. Therefore the linear theory will be used to evaluate the efficiency of energy transfer from a charged line pulse generator to a mismatched load through an exponential line.

Voltage and Current on an Exponential Transmission Line

On a transmission line which has the same velocity of propagation at all points along the line but which has a variable characteristic impedance, the voltage is described by the solutions to the differential equation:

$$\frac{d^2\overline{V}}{dx^2} - \frac{1}{Z_o}\frac{dZ_o}{dx}\frac{d\overline{V}}{dx} - L_\ell C_\ell p^2\overline{V} = 0$$

where $\overline{V}(x,p)$ is the Laplace transform of the voltage, $V(x,t)$. The initial voltage and current are presumed to be zero everywhere along the line. An exponential line is one in which the impedance varies as $Z_o(x) = Z_o(o)e^{-kx}$, and hence $Z_o^{-1}\, dZ_o/dx = -k$. The solution to the above equation as given in Lewis and Wells is

$$\begin{aligned}\overline{V}(x,p) &= \overline{V}+(x,p) + \overline{V}-(x,p)\\ &= \overline{V}+(o,p)e^{-\frac{k}{2}x}\, e^{-qx}\\ &\quad +\overline{V}-(o,p)e^{-\frac{k}{2}x}\, e^{qx} \qquad (1)\end{aligned}$$

where $\overline{V}+(o,p)$ and $\overline{V}-(o,p)$ are the transforms of the two voltage waves at x=0 which are propagated toward greater and toward smaller values of x respectively, and $q=T_o\sqrt{p^2+a^2}$, $T_o\sqrt{L_\ell C_\ell}$, $a = k/2T_o$. The corresponding current can be obtained from the relation $L_\ell p\overline{I} = -d\overline{V}/dx$.

$$\bar{I}(x,p) = \frac{\sqrt{p^2+a^2} + a}{p} \; \frac{\bar{V}+(x,p)}{Z_o(x)}$$

$$- \frac{\sqrt{p^2+a^2} - a}{p} \; \frac{\bar{V}-(x,p)}{Z_o(x)} \qquad (2)$$

Approximate Linear Theory

The factors e^{qx} and e^{-qx} appearing in the expression for $\bar{V}(x,p)$ are expanded as a series of powers of 1/p. For the linear approximation all terms with powers greater than first are neglected. The multiplicative factors appearing in the expression for $\bar{I}(x,p)$ are approximated in the same way. Therefore the approximate expressions for the voltage and current are:

$$\bar{V}(x,p) = \bar{V}+(x,p) + \bar{V}-(x,p)$$

$$= \bar{V}+(o,p)e^{-pT_o x} \left(1 - \frac{a^2 T_o x}{2p}\right) e^{-\frac{k}{2} x}$$

$$+ \bar{V}-(o,p) \; e^{pT_o x} \left(1 + \frac{a^2 T_o x}{2p}\right) e^{-\frac{k}{2} x} \qquad (3)$$

$$\bar{I}(x,p) = \left(1 + \frac{a}{p}\right) \frac{\bar{V}+(x,p)}{Z_o(x)} - \left(1 - \frac{a}{p}\right) \frac{\bar{V}-(x,p)}{Z_o(x)} \qquad (4)$$

The approximate linear theory will be used to calculate the waveform of the voltage across a load, Z_ℓ, at the end of a section of exponential line transformer. Z_ℓ will not necessarily be equal to the characteristic impedance at the end of the transformer. The input voltage to the transformer will be that generated by a charged section of constant impedance transmission line switched into the transformer line. To keep the impedance at the pulse generator high for fast switching the characteristic impedance of the exponential line at the input will be taken equal to that of the transmission line generator $Z_o(o)=Z_{oo}$. Since the motivation for this study is to examine the efficiency of energy transfer, the impedance of the switch will be assumed zero. This neglect of the energy loss due to the risetime of the input pulse seems justified here because this is an approximate calculation and because the transmission line transformers used in practice will be only approximately exponential. When the pulse forming line charged to V_o is switched into the transformer at x=o, only a + wave can be launched. The voltage across the output of the charged line must be equal to that across the input to the exponential line.

$$\frac{V_o}{p} - Z_{oo}\,\bar{I}(o,p) = \bar{V}+(o,p)$$

From Equation (4), the current is

$$\bar{I}(o,p) = \left(1 + \frac{a}{p}\right)\frac{V+(o,p)}{Z_{oo}}$$

Therefore

$$\overline{V}+(o,p) = \frac{V_o}{2p+a}$$

In the spirit of the linear approximation, this can be approximated by the first terms of its power series expansion

$$\overline{V}+(o,p) \approx \frac{V_o}{2p} \left(1 - \frac{a}{2p}\right) \qquad (5)$$

Now let the point x represent the end of the exponential line section. The voltage wave incident there is

$$\overline{V}+(x,p) = e^{-\frac{k}{2}x} e^{-pT_o x} \frac{V_o}{2p} \left(1 - \frac{a}{2p}\right)\left(1 - \frac{a^2 T_o x}{2p}\right) \qquad (6)$$

When this wave arrives at the load, Z_ℓ, load current flows and a reflected wave is also generated. The voltage across the end of the transformer must be that applied to the load:

$$\overline{V}+(x,p) + \overline{V}-(x,p) = \overline{V}_\ell(p),$$

and the line and load currents are equal:

$$\frac{1}{Z_o}\left[\left(1 + \frac{a}{p}\right)\overline{V}+(x) - \left(1 - \frac{a}{p}\right)\overline{V}-(x)\right] = \frac{\overline{V}_\ell}{Z_\ell}$$

where Z_o now stands for the characteristic impedance of the exponential line at the load end. The load voltage is then found in terms of the incident wave

$$\overline{V}_\ell = \frac{2\overline{V}+(x)}{1 + \frac{Z_o}{Z_\ell} - \frac{a}{p}}$$

or

$$\overline{V}_\ell \approx \frac{Z_\ell}{Z_\ell + Z_o} 2\overline{V}+(x) \left(1 + \frac{Z_\ell}{Z_\ell + Z_o} \frac{a}{p}\right) \qquad (7)$$

The expression for $\overline{V}+(x)$ from (6) can now be substituted into (7). The factor $e^{-pT_o x}$ represents the time delay due to propagation through the transformer. The factor $e^{-\frac{k}{2}x} = \sqrt{Z_o/Z_{oo}}$. If

$$V_1 = V_o \sqrt{\frac{Z_o}{Z_{oo}}} \frac{Z_\ell}{Z_\ell + Z_o}$$

then the transform of the load voltage is

$$\overline{V}_\ell = \frac{V_1}{p} \left(1 - \frac{a}{2p}\right) \left(1 - \frac{a^2 T_o x}{2p}\right) \left(1 + \frac{Z_\ell}{Z_\ell + Z_o} \frac{a}{p}\right)$$

If powers of p^{-1} greater than second are neglected this becomes

$$\overline{V}_\ell = \frac{V_1}{p} \left[1 - \frac{a}{p} \left(\frac{1}{2} + \frac{aT_o x}{2} - \frac{Z_\ell}{Z_\ell + Z_o}\right)\right]$$

$$= \frac{V_1}{p} \left[1 - \frac{\alpha}{p}\right]$$

and the inversion to the time domain gives $V_\ell(t) = V_1[1-\alpha t]$ where

$$\alpha = a\left(\frac{1}{2} + \frac{aT_o x}{2} - \frac{Z_\ell}{Z_\ell + Z_o}\right)$$

Let

$t_o = T_o x$ = transit time of the exponential line

$t_1 = a^{-1} = 2T_o/k$ = time for e-fold voltage transformation as pulse propagates along the exponential line.

then

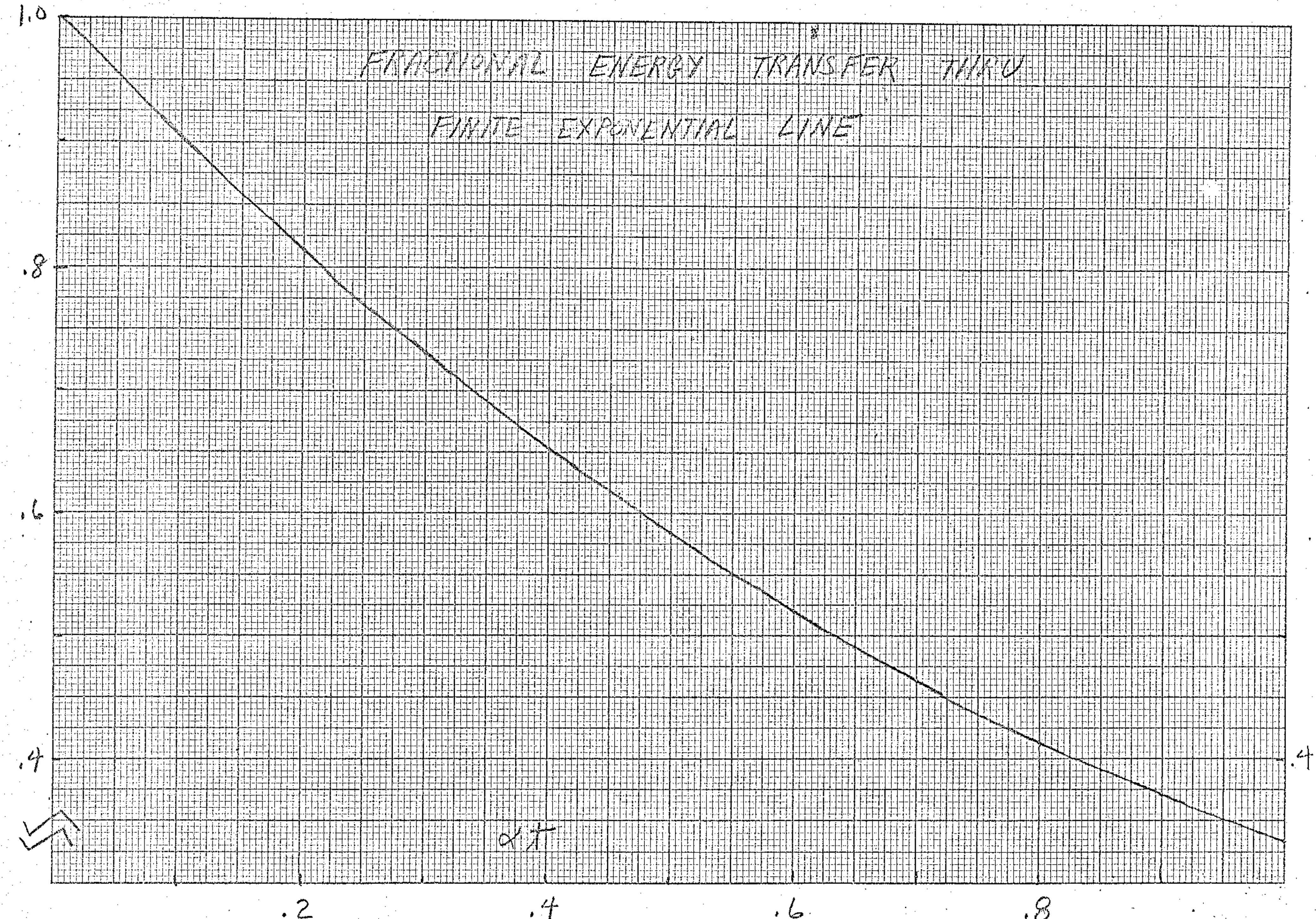
$$\alpha = \frac{1}{t_1}\left(\frac{1}{2} + \frac{t_o}{2t_1} - \frac{Z_\ell}{Z_\ell + Z_o}\right)$$

This approximate voltage waveform can now be used to estimate the energy delivered into a load during a time interval t. It is:

$$W_\ell = \int_0^t \frac{V_\ell^2}{Z_\ell}\, dt$$

$$= \frac{V_1^2}{Z_\ell} \int_0^t (1-\alpha t)^2\, dt$$

$$= \frac{V_1^2}{Z_\ell}\, t \left[1 - \alpha t + \frac{(\alpha t)^2}{3}\right]$$


FRACTIONAL ENERGY TRANSFER THRU
FINITE EXPONENTIAL LINE
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